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ACOUSTIC PRESSURE OF A FREELY VIBRATING CIRCULAR PLATE WITHOUT BAFFLE

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Formula for an acoustic pressure of a circular plate under free vibrations without baffle board is derived with the use of oblate spheroidal coordinate system. The result is obtained in terms of a single series of spheroidal function products. The number of terms ensuring a required accuracy can be determined numerically. The field radiated by a plate without baffle is analysed for the first three vibration modes on the basis of their directional characteristics and accounting for various values of an interference parameter $h = 2\pi a/\lambda$.

1. Introduction

The knowledge of basic quantities that characterize an acoustic field is necessary to employ plates and shells vibrating systems used acoustic diagnostic applicances as well as receivers and transmitters of acoustic waves. It was not earlier than in the eighties that detailed analytical investigations on the acoustic field radiated by a circular plate begun Analysis comprised free vibrations [10, 11] and forced vibrations [12]. Damping effects and modifications of wave emission by its specific field were also accounted for [8, 13]. Relevant phenomena were assumed to be linear and vary in time in a sinusoidal manner. Relatively simple mathematical tools were used since a plate was considered to vibrate in an infinite plane baffle. No such baffle board exist in real situations and the obtained results were valid for sufficiently high frequencies only.

Acoustic fields around sources without baffles or supplied with finite rigid baffles were analysed in [1–5]. Directional characteristics and impedances for pulsating and oscillating piston with uniform wave velocity distribution were found by solving a wave equation with the use of separation of variables in the spheroidal frame of reference.

To date, investigations have been taking on acoustic fields radiated by a circular plate without any baffle or with a finite baffle. This subject is dealt with in this paper, which is an extension of [7] and also refers to [1–5].

Properties of the oblate spheroidal coordinate system are used to derive a formula for acoustic pressure of a circular plate freely vibrating without a baffle. The plate is assumed to be thin, homogeneous and clamped at the circumference; surrounding medium is lossless. Employing the known solution for free vibration of such a plate, wave velocity distribution is found and transformed to the oblate spheroidal frame of reference. Such a degeneration of the frame leads to a formula for an acoustic pressure in terms of series of spheroidal function products. Since no standard numerical procedures have been worked out to calculate values of spheroidal functions, an attempt is made to prepare suitable algorithms. To determine eigenvalues of the wave equation and the expansion coefficients d_r^{mn}, Hodge's method [6] is used. Angular and radical spheroidal functions and the necessary derivatives of radial functions are calculated with the help of recurrence relationships given by FLAMMER [15], to within an accuracy of 6–7 significant figure after decimal point.

Directional characteristics are given for the first three modes of freely vibrating circular plate without baffle and – for the source of comparison – with finite baffle as well as for a piston with uniform vibration velocity distribution are also presented. For wave lengths shorter than the dimensions of considered sources the obtained results fully agree with the characteristics calculated for a freely vibrating plate with an infinite baffle by using Huygens–Rayleigh integral [10].

2. Vibration equation for a plate

Free vibrations of a thin homogeneous plate of density ρ and thickness H, small compared with its diameter $2a$, is described by an equation [16]

$$\frac{\partial^2 w}{\partial t^2} + \frac{B}{M}\nabla^4 w = 0, \tag{2. 1}$$

where M is a plate mass per unit area, B denotes its flexural stiffness and w is a deflection function. For a circular plate the equation (2. 1) is solved in polar coordinates and the deflections are [16]

$$w(r,t) = w(r)e^{-i\omega t} = [A_0 J_0(kr) + B_0 I_0(kr)]e^{-i\omega t} \tag{2. 2}$$

where

$$k^2 = \omega\sqrt{M/B}, \tag{2. 2a}$$

ω – frequency, $A_0 B_0$ – constants, J_0 – Bessel's zero-degree function of the first kind, I_0 – modified Bessel's zero-degree function of the first kind.

The vibration process is harmonic hence the equation (2.2) supplies the following expression for vibration velocity

$$v(r,t) = v(r)e^{-i\omega t} \tag{2.3}$$

$$v(r) = AJ_0(kr) + BI_0(kr) \tag{2. 4}$$

where

$$A = -i\omega A_0, \quad B = -i\omega B_0 \tag{2. 4a}$$

The velocity satisfies the boundary conditions for a clamped plate :

$$v(r)\big|_{r=a} = 0 \tag{2.5}$$

$$\left.\frac{dv(r)}{dr}\right|_{r=a} = 0 \tag{2.6}$$

Their use in (2.4) leads to the so-called frequency equation

$$J_0(ka)I_1(ka) + J_1(ka)I_0(ka) = 0, \tag{2.7}$$

whose solution is a series $k = k_l$ for $l = 1, 2, 3$.

On account of the formula (2.2a) the free vibration frequency for the $(0, l)$ mode is

$$f_l = k_l^2\sqrt{M/B}/2\pi \tag{2.8}$$

Formula (2.4) becomes

$$\frac{v_l(r)}{A} = J_0(k_l r) - \frac{J_0(k_l a)}{I_0(k_l a)} I_0(k_l r) \tag{2.9}$$

3. Transformation of velocity distribution for the OSCS

The equation (2.9) will now be expressed in the oblate spheroidal coordinate system OSCS with the use of the following transformation

$$\begin{aligned} x &= b\,[(1-\eta^2)(\xi_0^2+1)]^{1/2}\cos\varphi \\ y &= b\,[(1-\eta^2)(\xi_0^2+1)]^{1/2}\sin\varphi \\ z &= b\,\xi\eta \end{aligned} \tag{3.1}$$

where

$$\varphi \in \langle 0, 2\pi \rangle,\ \eta \in \langle -1, 1 \rangle,\ \xi \in \langle 0, \infty \rangle \tag{3.1a}$$

Due to the symmetry with respect to the z–axis, Fig. 1, the problem can be considered in the xz–plane by assuming $\varphi = 0$ in the formulae (3.1). Since $r^2 = x^2 + y^2$ on account of (3.1) we get

$$r = b\,[(1-\eta^2)(\xi_0^2+1)]^{1/2} \tag{3.2}$$

Denoting the surface of a spheroid on which the source is transformed by ξ_0 and substituting (3.2) into (2.9), the following formula for the vibration velocity is reached in the OSCS

$$\frac{v_l(\eta, \xi_0)}{A} = \left[J_0\left(k_l b\sqrt{(1-\eta^2)(\xi_0^2+1)}\right) - \frac{J_0(k_l a)}{I_0(k_l a)} I_0\left(k_l b\sqrt{(1-\eta^2)(\xi_0^2+1)}\right) \right] \tag{3.3}$$

where $2b$ is a distance between focal points.

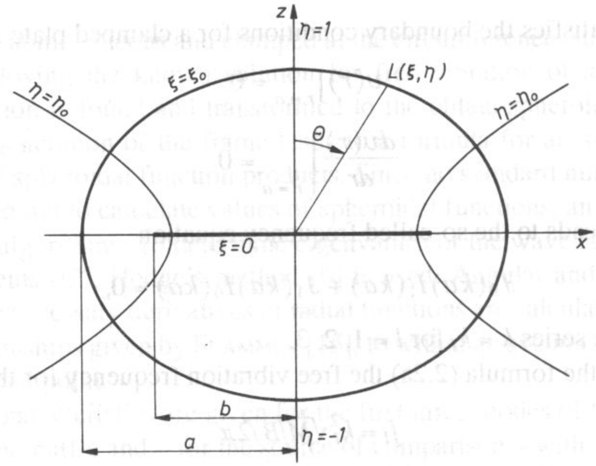

FIG. 1. Circular plate in the oblate spheroidal coordinate system.

4. Solution of Helmholtz equation in the OSCS

Let $\Phi(\xi, \eta, \varphi, t)$ denote a potential of a velocity field radiated by a plate. For harmonic processes can be expressed as $\Phi(\xi, \eta, \varphi, t)$

$$\Phi(\xi, \eta, \varphi, t) = \Psi(\xi, \eta, \varphi)\, e^{-i\omega t} \tag{4. 1}$$

In order to determine a distribution of the acoustic field around a considered source, Neuman's boundary value problem for the Helmholtz equation should be solved in the OSCS

$$\left[\frac{\partial}{\partial n}(1-\eta^2)\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}(\xi^2+1)\frac{\partial}{\partial \xi} + \frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)}\frac{\partial^2}{\partial \varphi^2} + h(\xi^2+\eta^2)\right]\Psi(\eta, \xi, \varphi) = 0 \tag{4. 2}$$

where $h = k_0 b$ – dimensionless wave number ($k_0 = 2\pi/\lambda$). Boundary condition on the surface of a chosen spheroid $\xi = \xi_0$ has the form

$$\frac{\partial \Psi}{\partial n} = \frac{1}{h_\xi}\frac{\partial \Psi}{\partial \xi}\bigg|_{\xi=\xi_0} = \begin{cases} -\upsilon(\eta, \xi_0), & 0 \le \eta \le 1 \\ 0 & -1 \le \eta < 0 \end{cases} \tag{4. 3}$$

where $\upsilon(\eta, \xi_0)$ – according to the formula (3. 3)

$$h_\xi\big|_{\xi=\xi_0} = h_{\xi_0} = b\left(\frac{\xi_0^2+\eta^2}{\xi_0^2+1}\right)^{1/2}. \text{ – the so-called scaling factor [15]} \tag{4. 4}$$

In the OSCS the equation (4.2) can be separated into two differential equations, each being satisfied by its eigenfunction that depends on only one spatial variable – ξ or η.

Due to the symmetry of radiated waves with respect to the z–axis ($\varphi = 0$), the solution for outgoing waves is assumed to be a superposition of eigenfunctions, namely

$$\Psi(\eta, \xi) = \sum_{n}^{\infty} A_n S_{on}^{(1)}(- ih, \eta) R_{on}^{(3)}(- ih, i\xi) \tag{4.5}$$

where A_n constant coefficients to be obtained from boundary conditions, $S_{on}^{(1)}(- ih, \eta)$ – angular spheroidal function of the first kind, $R_{on}^{(3)}(- ih, i\xi)$ – radial spheroidal function of the third kind depending on the distance of a wave from its source. Asymptotic properties of radial functions [15],

$$R_{on}^{(3)}(- ih, i\xi) \xrightarrow{\xi \longrightarrow \infty} (- i)^{n+1} \frac{e^{ih\xi_\infty}}{h\xi_\infty} \tag{4.6}$$

are such that the assumed solution (4.5) does also satisfy Somerfield's conditions.

5. Determination of coefficients A_n

To find A_n that appear in (4.5), the known function $\upsilon(\eta, \xi_0)$ describing vibration velocity of a source on the surface of a selected spheroid $\xi = \xi_0$, is expanded into a series with respect to the angular spheroidal functions [17]

$$\upsilon(\eta, \xi_0) = \frac{1}{h_{\xi_0}} \sum_{n}^{\infty} V_n S_{on}(- ih, \eta) \tag{5.1}$$

In turn, to determine the expansion coefficients V_n, the series (5.1) is multiplied by an orthogonal spheroidal function $S_{on'}(- ih, \eta)$ and integrated on the surface of spheroid. The following expression for V_n is arrived at:

$$V_n = \frac{1}{N_{on}(-ih)} \int_0^1 h_{\xi_0} \upsilon(\eta, \xi_0) S_{on}(-ih, \eta) d\eta \tag{5.2}$$

in which the $N_{on}(-ih)$ has the form

$$N_{on}(-ih)\, \delta_{nn'} = \int_{-1}^{1} S_{on}(-ih, \eta) S_{on'}(-ih, \eta) d\eta \tag{5.3}$$

On account of the condition (4.3), we get

$$\upsilon(\eta, \xi_0) = -\frac{1}{h_{\xi_0}} \sum_{n}^{\infty} A_n S_{on}(-ih, \eta) \frac{\partial R_{on}^{(3)}(-ih, \xi_0)}{\partial \xi} \tag{5.4}$$

Equating the corresponding terms in the series (5.1) and (5.4), we finally obtain

$$A_n = \frac{1}{N_{on}(-ih) R_{on}^{(3)'}(-ih, \xi_0)} \int_0^1 h_{\xi_0} \upsilon(\eta, \xi_0) S_{on}(-ih, \eta) d\eta \tag{5.5}$$

where

$$R_{on}^{(3)'}(-ih, \xi_0) = \frac{\partial R_{on}^{(3)}(-ih, \xi_0)}{\partial \xi} \tag{5.6}$$

6. Acoustic field of a circular plate

To describe an acoustic field radiated by a plane circular plate with no baffle the following acoustic pressure is applied

$$p = \rho \frac{\partial}{\partial t} \Phi(\xi, \eta, \varphi, t) = -i\rho \mathrm{ch}\, \Psi(\xi, \eta) \tag{6.1}$$

where ρ – density of medium, c – sound velocity. The formula (6.1) together with (4.5) to describe $\Psi(\xi, \eta)$ refers to a source on a spheroid ξ_0. To make this solution valid for the considered plate, the coordinate system must be degenerated by assuming $\xi_0 = 0$. We get

$$p(\xi, \eta) = -i\rho h \sum_n A_n S_{on}(-ih, \eta) R_{on}^{(3)}(-ih, i\xi) \tag{6.2}$$

where

$$A_n = \frac{1}{N_{on}(-ih)} \frac{1}{R_{on}^{(3)'}(-ih, 0)} \int_0^1 v_l(\eta) S_{on}(-ih, \eta)\, \eta\, d\eta \tag{6.3}$$

and

$$v_l(\eta) = A\left[J_0(k_l a\sqrt{1-\eta^2}) - \frac{J_0(k_l a)}{I_0(k_l a)} I_0(k_l a\sqrt{1-\eta^2})\right] \tag{6.4}$$

7. Diagrams and conclusions

When analysis an acoustic field radiated by a circular plate without a baffle, two cases can be examined each depending on a manner in which the waves are emitted by a system with the plate as a source of vibrations.

Model 1. A field radiated by the upper surface of the plate. The source should be assumed on the upper surface of spheroid and the field is to be calculated according to (6.2).

Model 2. A field radiated by both upper and lower surfaces of the plate. In the absence of baffle this results in two axially located sources on the upper and lower surfaces and vibrating in the counterphase manner. The coefficients A_n (6.3) should now be calculated from

$$A_n^* = \frac{-b}{N_{on}(-ih) R_{on}^{(3)'}(-ih, 0)} \left[\int_0^1 v_L(\eta) S_{on}(-ih, \eta)\, \eta\, d\eta + \int_{-1}^0 v_L(\eta) S_{on}(-ih, \eta)\, \eta\, d\eta\right] \tag{7.1}$$

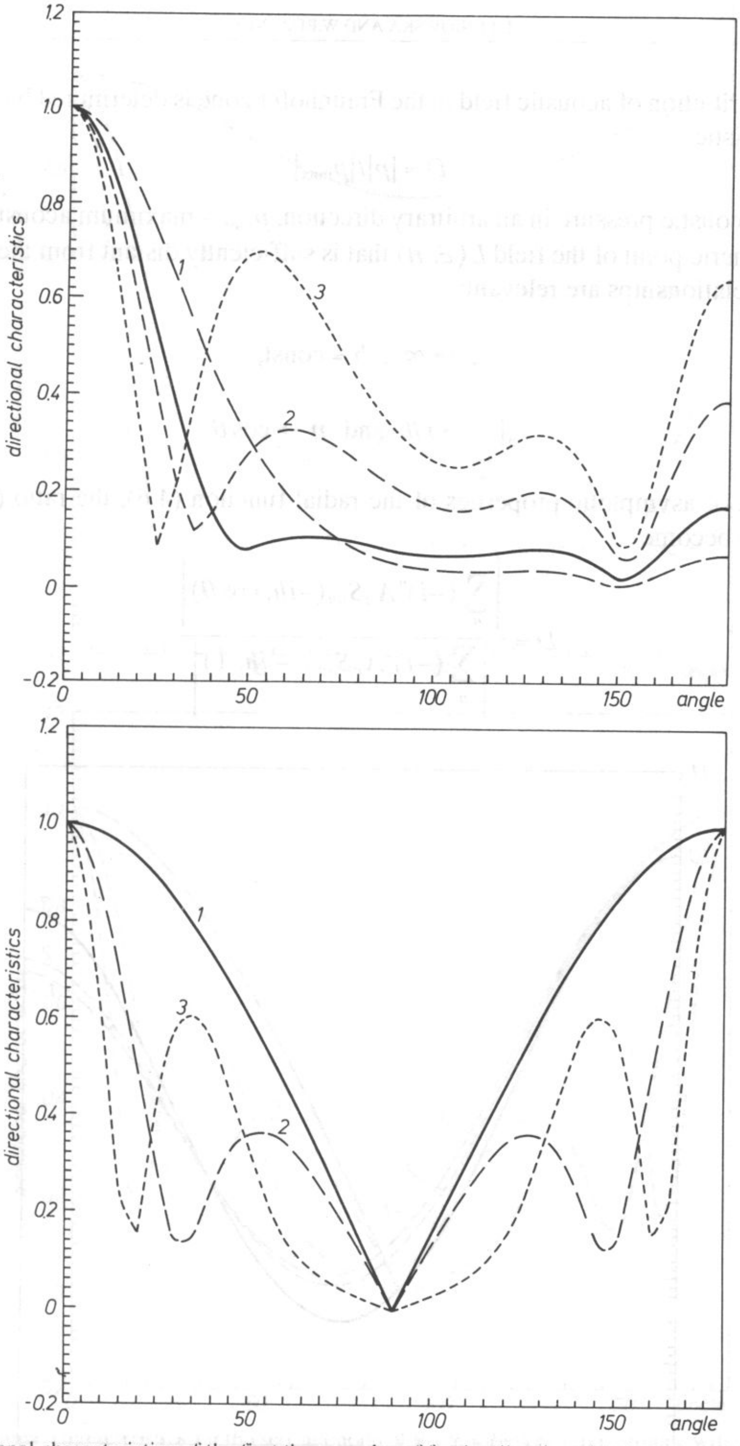

FIG. 2. Directional characteristics of the first three modes of freely vibrating circular plate without a baffle for $h = 5$; *a* – model 1, *b* – model 2. Curves are numbered according to the 1st, 2nd and 3rd mode.

The distribution of acoustic field in the Fraunhofer zone is determined by a directional characteristic

$$D = |p|/|p_{\max}| \tag{7.2}$$

where p – acoustic pressure in an arbitrary direction, $p_{\max}$ – maximum acoustic pressure.

At a generic point of the field $L(\xi, \eta)$ that is sufficiently distant from the source, the following relationships are relevant:
if

$$\xi \to \infty\ ,\ \ b = \text{const}, \tag{7.3}$$

then

$$\xi\Big|_{\xi_\infty} \to r/b \ \text{ and } \ \eta \to \cos\theta \tag{7.4}$$

On account of asymptotic properties of the radial function (4.6), the ratio (7.2) can be rewritten to become

$$D = \frac{\left|\sum\limits_n (-i)^n A_n S_{on}(-ih, \cos\theta)\right|}{\left|\sum\limits_n (-i)^n A_n S_{on}(-ih, 1)\right|} \tag{7.5}$$

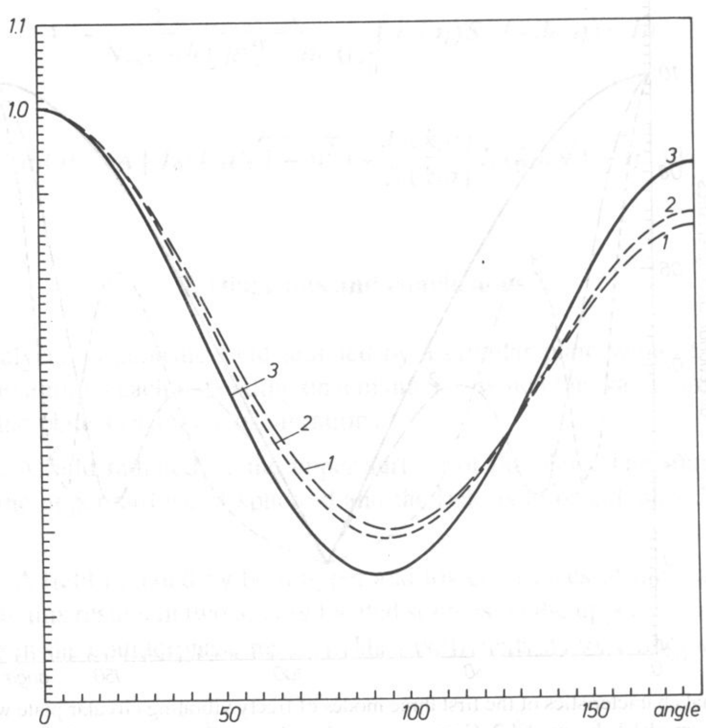

FIG. 3. Directional characteristics of the first three modes of freely vibrating circular plate without a baffle for $h = 1$, model 1. Curves are numbered according to the 1st, 2nd and 3rd mode.

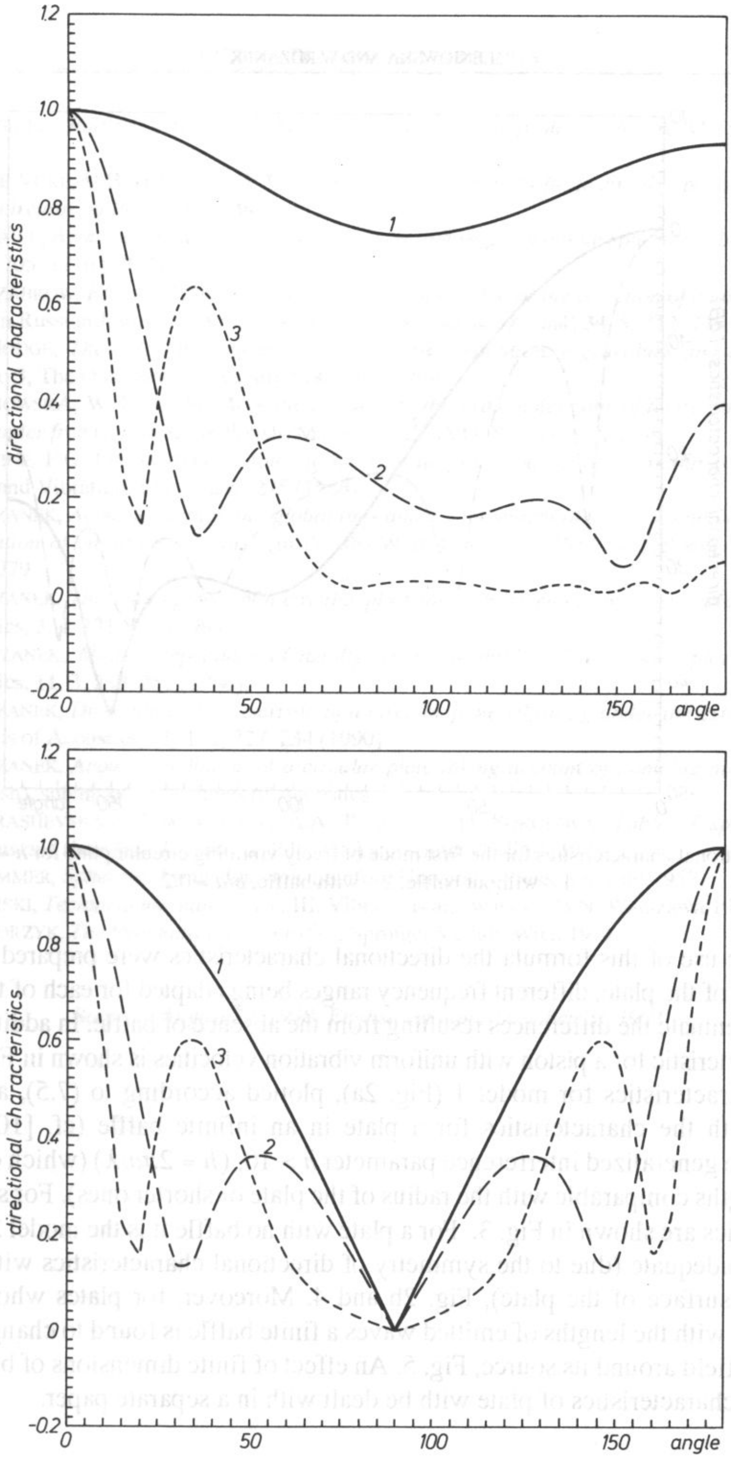

FIG. 4. Directional characteristics for the second mode of freely vibrating circular plate without a baffle for various values of *h*; *a* – model 1, *b* – model 2, 1: *h* = 1, 2: *h* = 5, 3: *h* = 10.

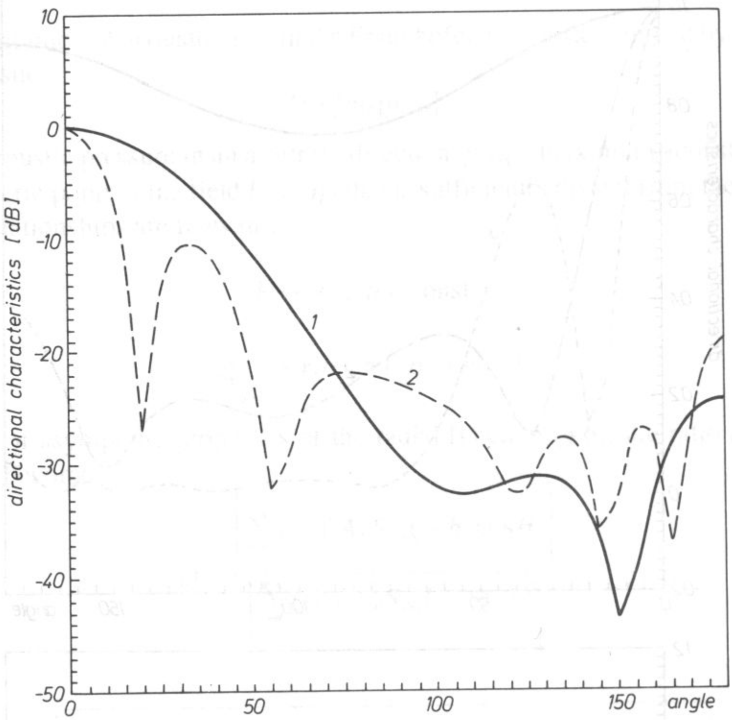

FIG. 5. Directional characteristics for the first mode of freely vibrating circular plate for $h = 5$, model 1. 1 – without baffle, 2 – with baffle, $b/a = 0.2$.

With the use of this formula the directional characteristics were prepared for the first three modes of the plate, different frequency ranges being adapted for each of the modes in order to accentuate the differences resulting from the absence of baffle. In addition, a directional characteristic for a piston with uniform vibration velocities is shown in Fig. 2.

The characteristics for model 1 (Fig. 2a), plotted according to (7.5), are found to coincide with the characteristics for a plate in an infinite baffle (cf. [10, Fig. 15]), provided the generalized interference parameter $h > 10$, ($h = 2\pi a/\lambda$) (which corresponds to wavelengths comparable with the radius of the plate or shorter ones). For smaller h the characteristics are shown in Fig. 3. For a plate with no baffle it is the model 2 that seems to be more adequate (due to the symmetry of directional characteristics with respect to the middle surface of the plate), Fig. 2b and 4. Moreover, for plates whose radius is comparable with the lengths of emitted waves a finite baffle is found to change the shape of acoustic field around its source, Fig. 5. An effect of finite dimensions of baffles on the directional characteristics of plate with be dealt with in a separate paper.

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