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Research Paper

Dispersion Curves of Love Waves in Elastic Waveguides Loaded with a Newtonian Liquid Layer of Finite Thickness

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In this paper, the authors analyse the propagation of surface Love waves in an elastic layered waveguide (elastic guiding layer deposited on an elastic substrate) covered on its surface with a Newtonian liquid layer of finite thickness. By solving the equations of motion in the constituent regions (elastic substrate, elastic surface layer and Newtonian liquid) and imposing the appropriate boundary conditions, the authors established an analytical form of the complex dispersion equation for Love surface waves. Further, decomposition of the complex dispersion equation into its real and imaginary part, enabled for evaluation of the phase velocity and attenuation dispersion curves of the Love wave. Subsequently, the influence of the finite thickness of a Newtonian liquid on the dispersion curves was evaluated. Theoretical (numerical) analysis shows that when the thickness of the Newtonian liquid layer exceeds approximately four penetration depths 4δ of the wave in a Newtonian liquid, then this Newtonian liquid layer can be regarded as a semi-infinite half-space. The results obtained in this paper can be important in the design and optimization of ultrasonic Love wave sensors such as: biosensors, chemosensors and viscosity sensors. Love wave viscosity sensors can be used to assess the viscosity of various liquids, e.g. liquid polymers.

Keywords: Love waves; ultrasonic sensors; Newtonian liquid; penetration depth; biosensors; chemosensors; viscosity sensors.

1. Introduction

Ultrasonic bulk and surface waves (Achenbach, 1973; Auld, 1990; Royer, Dieulesaint, 2000; Rose, 2014) are widely used to develop sensors of physical properties of materials in NDT, biosensors and chemosensors (Ballantine *et al.*, 1997; Qian *et al.*, 2010; Rocha Gaso *et al.*, 2013; Liu, 2014; Kiełczyński *et al.*, 2014a; 2014b; 2014c; Hong *et al.*, 2014; Goto *et al.*, 2015; Wang *et al.*, 2015). Love waves are the most promising surface waves in applications for sensors working in a liquid environment (Achenbach, 1973; Royer, Dieulesaint, 2000; Rose, 2014).

Surface waves of the Love type have a number of unique features, what differentiates them from other types of waves. Firstly, Love surface waves have only one shear horizontal SH component of vibrations. As a result, Love waves are slightly affected by loading with a viscous liquid and consequently they may propagate long distances without a significant attenuation. Secondly, the energy of Love waves attains high densities in a guiding surface layer, deposited on the semi-infinite substrate. This property is crucial in development of Love wave sensors working usually in a contact with liquid environment (investigated analyte). Thirdly, the analytical formulas describing propagation of Love waves are relatively compact, what enables drawing clear physical conclusions.

The main advantage of Love wave sensors is their ability to operate in a liquid environment. This results from the fact that the transverse component of the mechanical displacement of the Love wave generates, in the liquid loading the waveguide surface, a bulk transverse wave. This wave penetrates into the liquid at a small distance (penetration depth) and quickly attenuates with increasing distance from the surface of the waveguide.

The ability to operate in a liquid environment is one of the most important features of any Love wave biosensor. In fact, Love wave biosensors extract the investigated analyte from the surrounding liquid, forming a so called coating layer, which properties can be correlated with phase velocity and attenuation of Love surface waves.

In this paper we will determine how thick the layer of a loading viscous liquid should be to be considered as a layer of an infinite thickness. The knowledge of this limiting thickness is very important in a design of stable and accurate Love wave biosensors.

To date, the theory of Love wave propagation has been developed for elastic waveguide structures loaded on the surface with the semi-infinite layer of Newtonian liquid (Kiełczyński *et al.*, 2012). In the present work, we investigate a more realistic (general) situation, i.e. we consider a case in which the layer of a Newtonian liquid has finite thickness. To this end, the authors formulated and solved the coupled differential problem (Sturm-Liouville Problem), which describes the propagation of the Love wave in the analyzed waveguide structure shown in Fig. 1.

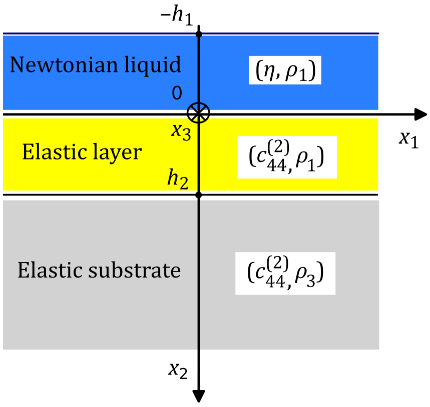

Fig. 1. Cross-section of the analyzed lossless Love wave waveguide loaded with a lossy Newtonian liquid. Love surface waves propagate along the x_1 axis. Shear horizontal (SH) mechanical displacement u_3 of the Love wave is directed along the x_3 axis.

The influence of thickness of the layer of a Newtonian liquid on the dispersion curves (phase velocity and attenuation) of the Love wave has been estimated. The results of this work may find application in the design and optimization of ultrasonic Love wave sensors such as: biosensors, chemosensors and viscosity sensors.

Love wave viscosity sensors can be employed to evaluate the viscosity of various liquids (e.g. liquid polymers) processed in the chemical and plastics industries. Furthermore, the analysis performed in this study can be employed in nondestructive (NDT) investigations of composite materials in order to process the obtained experimental data.

2. Structure and material parameters of the surface Love wave waveguide

The propagation of Love surface waves in layered waveguides, given their material and geometric parameters, can be formulated in terms of the Direct Sturm-Liouville Problem (Kiełczyński, 2018). A solution to this Direct Sturm-Liouville Problem is in a form of discrete eigenvalue-eigenvector pairs. An eigenvalue determines the complex wave vector (i.e. the phase velocity and attenuation) of the Love wave and the corresponding eigenvector is a function describing distribution of the mechanical displacement of the Love as a function of depth, i.e. the distance from the guiding surface.

Since, the layered waveguide structure is lossy (Newtonian liquid), the wave number k of the Love wave is a complex quantity:

$$k = k_0 + j\alpha, \tag{1}$$

where: $j = \sqrt{-1}$ is the imaginary unit.

The real part (k_0) of the wave number k determines the phase velocity of the Love wave: $v_p = \omega / k_0$, where: ω is the angular frequency of the wave, and the imaginary part α is the coefficient of attenuation of the Love wave.

2.1. Geometry and material parameters of the lossy Love wave waveguide

The layered elastic composite waveguide analyzed in this paper is shown in Fig. 1. The waveguide is designed to support shear horizontal (SH) surface waves of the Love type. The composite waveguide consists of a lossless elastic surface layer ($h_2 > x_2 > 0$), which is rigidly bonded to a lossless infinite elastic substrate occupying the lower half-space ($x_2 > h_2$). In addition, top of the surface layer of the waveguide ($x_2 = 0$) is loaded with a layer of viscous (Newtonian) liquid ($0 > x_2 > -h_1$) of finite thickness h_1. The Newtonian liquid is characterized by its viscosity η and density ρ_1 as well as by the quasi shear modulus $c_{44}^{(1)} = -j\omega\eta$.

The surface layer is a lossless elastic material with a real shear modulus of elasticity $c_{44}^{(2)}$, such as gold (Au). The elastic substrate is a semi-infinite elastic medium with a real shear modulus of elasticity equal to $c_{44}^{(3)}$, such as ST-cut Quartz material supporting pure SH bulk waves. The x_2 axis is directed into the bulk of the substrate. All material parameters of the composite waveguide change only along the x_2 axis but are homogeneous along the x_1 and x_3 axes. Love surface waves have only one non-zero shear-horizontal (SH) component of the mechanical displacement u_3, which is directed along the x_3 axis, parallel to the surface ($x_2 = 0$) of the waveguide and perpendicular to the direction of the Love wave propagation along the x_1 axis.

3. Mathematical statement of the problem

3.1. Equations of motion

3.1.1. Lossy layer $(-h_1 < x_2 < 0)$ of a Newtonian liquid with a finite thickness h_1

The mechanical displacement of the Love wave inside the lossy layer $(-h_1 < x_2 < 0)$ of a Newtonian liquid, of finite thickness h_1, is in the form of a bulk transverse wave. The mechanical displacement $u_3^{(1)}$ of the bulk SH wave is governed by the following Navier-Stokes partial differential equation (Kiełczyński *et al.*, 2012):

$$\frac{\partial^2 u_3^{(1)}}{\partial t^2} = \left(\frac{-j\omega\eta}{\rho_1}\right)\left(\frac{\partial^2 u_3^{(1)}}{\partial x_1^2} + \frac{\partial^2 u_3^{(1)}}{\partial x_2^2}\right), \tag{2}$$

where ρ_1 is the liquid density, η its viscosity and ω is the angular frequency of the Love surface wave.

3.1.2. Lossless elastic surface layer $(h_2 > x_2 > 0)$

The mechanical displacement $u_3^{(2)}$ of the Love wave in the elastic surface layer is governed by the following equation of motion (Kiełczyński *et al.*, 2012):

$$\frac{1}{v_2^2}\frac{\partial^2 u_3^{(2)}}{\partial t^2} = \frac{\partial^2 u_3^{(2)}}{\partial x_1^2} + \frac{\partial^2 u_3^{(2)}}{\partial x_2^2}, \tag{3}$$

where $v_2 = \left(c_{44}^{(2)}/\rho_2\right)^{1/2}$ is bulk SH wave velocity in the lossless surface layer, $c_{44}^{(2)}$ its storage modulus of elasticity, ρ_2 is the density of the lossless elastic surface layer. Since $c_{44}^{(2)}$ and ρ_2 are real, the phase velocity v_2 is a real quantity as well.

3.1.3. Semi-infinitive elastic substrate $(x_2 > h_2)$

The mechanical displacement $u_3^{(3)}$ of the Love wave in the elastic substrate satisfies the following partial differential equation (equation of motion) (Kiełczyński *et al.*, 2012):

$$\frac{1}{v_3^2}\frac{\partial^2 u_3^{(3)}}{\partial t^2} = \frac{\partial^2 u_3^{(3)}}{\partial x_1^2} + \frac{\partial^2 u_3^{(3)}}{\partial x_2^2}, \tag{4}$$

where $v_3 = \left(c_{44}^{(3)}/\rho_3\right)^{1/2}$ is the velocity of the bulk SH wave in the elastic substrate, $c_{44}^{(3)}$ its storage modulus of elasticity, and ρ_3 is the density in the elastic substrate. Since $c_{44}^{(3)}$ and ρ_3 are real, the phase velocity v_3 is a real quantity as well.

3.2. Analytical solutions

A general form of the solution for the equations of motion (Eqs (2)–(4)) corresponding to a time-harmonic Love surface wave is sought in the following form:

$$u_3(x_1, x_2, t) = f(x_2) \cdot \exp\left[j(kx_1 - \omega t)\right], \tag{5}$$

where $f(x_2)$ is the transverse distribution of the mechanical displacement u_3 of the Love surface wave as a function of depth x_2. The angular frequency is denoted by ω. The complex wave number k of the Love wave is given by $k = k_0 + j\alpha$, where k_0 determines the phase velocity $v_p = \omega/k_0$, and α is the wave attenuation.

3.2.1. Finite thickness layer of a Newtonian liquid $(0 > x_2 > -h_1)$

According to Eq. (5) the mechanical displacement $u_3^{(1)}$ in the finite thickness layer of a Newtonian liquid, satisfying Eq. (2), can be written as:

$$u_3^{(1)} = U(x_2) \cdot \exp\left[j\left(k \cdot x_1 - \omega t\right)\right]. \tag{6}$$

Substitution of Eq. (6) into Eq. (2) gives rise to the following Helmholtz type ordinary differential equation of the second order:

$$U''(x_2) + (k_1^2 - k^2) \cdot U(x_2) = 0, \tag{7}$$

where $k_1^2 = \frac{j\omega\rho_1}{\eta}$ is the complex bulk SH wave number in the loading Newtonian liquid, the complex wave number k of the Love surface wave is given by Eq. (1), ρ_1 is the liquid density, and η its viscosity.

We will seek the solution to Eq. (7) in the following form:

$$U(x_2) = C_1 \cdot \sin(q_1 \cdot x_2) + C_2 \cdot \cos(q_1 \cdot x_2), \tag{8}$$

where the complex transverse wave number is equal to $q_1 = (k_1^2 - k^2)^{1/2}$, and C_1 and C_2 are arbitrary constants.

The shear stress component $\tau_{23}^{(1)}$ associated with the mechanical displacement $u_3^{(1)}$ is given by:

$$\tau_{23}^{(1)} = c_{44}^{(1)}\frac{\partial u_3^{(1)}}{\partial x_2} = \Big(C_1 \cdot c_{44}^{(1)} \cdot q_1 \cdot \cos(q_1 \cdot x_2) - C_2 \cdot c_{44}^{(1)} \cdot q_1 \cdot \sin(q_1 \cdot x_2)\Big)\exp\left[j(kx_1 - \omega t)\right]. \tag{9}$$

The shear stress $\tau_{23}^{(1)}$ will enter into the appropriate boundary conditions at interfaces $x_2 = -h_1$ and $x_2 = 0$ (see Eqs (21) and (23) in Subsec. 3.4).

3.2.2. Lossless elastic surface layer $(h_2 > x_2 > 0)$

According to Eq. (5) the mechanical displacement $u_3^{(2)}$ in the elastic surface layer, satisfying Eq. (3), can be written as:

$$u_3^{(3)} = V(x_2) \cdot \exp\left[j(k \cdot x_1 - \omega t)\right], \tag{10}$$

where $V(x_2)$ is the transverse distribution of the mechanical displacement of the Love wave in the surface elastic layer.

Substitution of Eq. (10) into Eq. (3) results in:

$$V''(x_2) + (k_2^2 - k^2) \cdot V(x_2) = 0, \tag{11}$$

where the superscript prime $'$ denotes the differentiation with respect to the spatial variable x_2.

The solution to Eq. (11) can be expressed as:

$$V(x_2) = C_3 \cdot \sin(q_2 \cdot x_2) + C_4 \cdot \cos(q_2 \cdot x_2), \tag{12}$$

where the transverse wave number $q_2 = (k_2^2 - k^2)^{1/2}$, the bulk wavenumber in the elastic surface layer $k_2 = \frac{\omega}{v_2}$, and C_3 and C_4 are arbitrary constants.

The shear stress component $\tau_{23}^{(2)}$ that will be used later in boundary conditions is given by:

$$\tau_{23}^{(2)} = c_{44}^{(2)} \frac{\partial u_3^{(2)}}{\partial x_2} = \Big(C_3 \cdot c_{44}^{(2)} \cdot q_2 \cdot \cos(q_2 \cdot x_2) - C_4 \cdot c_{44}^{(2)} \cdot q_2 \cdot \sin(q_2 \cdot x_2)\Big) \exp[j(kx_1 - \omega t)]. \tag{13}$$

3.2.3. Semi-infinitive elastic substrate ($x_2 > h_2$)

According to Eq. (5) the solution of Eq. (4) for the mechanical displacement $u_3^{(3)}$ of the Love wave in the elastic substrate can be written as:

$$u_3^{(3)} = W(x_2) \cdot \exp[j(k \cdot x_1 - \omega t)], \tag{14}$$

where $W(x_2)$ is the transverse distribution of the mechanical displacement $u_3^{(3)}$ of the Love wave in the elastic substrate in the direction of axis x_2.

Substituting Eq. (14) into Eq. (4) yields:

$$W''(x_2) - (k^2 - k_3^2) \cdot W(x_2) = 0. \tag{15}$$

Since the amplitude of Love surface waves must tend to zero for $x_2 \to \infty$, we chose as a solution of Eq. (15) the following exponential expression:

$$W(x_2) = C_5 \cdot \exp(-b \cdot x_2), \tag{16}$$

where the transverse wave number is $b = (k^2 - k_3^2)^{1/2}$ and $\mathrm{Re}(b) > 0$. The bulk wavenumber is $k_3 = \frac{\omega}{v_3}$, and C_5 is an arbitrary constant.

The shear stress component $\tau_{23}^{(3)}$ required later in the boundary condition is given correspondingly by:

$$\tau_{23}^{(3)} = c_{44}^{(3)} \frac{\partial u_3^{(3)}}{\partial x_2} = C_5 \cdot c_{44}^{(3)} \cdot (-b) \cdot \exp(-b \cdot x_2) \cdot \exp[j(kx_1 - \omega t)]. \tag{17}$$

3.3. Propagation of Love surface waves in lossless elastic waveguides loaded with a lossy Newtonian liquid as a Direct Sturm-Liouville Problem

A set of 3 functions $\{U(x_2), V(x_2), W(x_2)\}$, given respectively by Eqs (8), (12), and (16) in Subsec. 3.2, which correspond to the transverse distribution of the mechanical displacement of the Love wave as a function of depth x_2, can be written in the following form:

$$\mathbf{f}(x_2) = \begin{bmatrix} U(x_2) \\ V(x_2) \\ W(x_2) \end{bmatrix}. \tag{18}$$

Three Helmholtz differential equations given by Eqs (7), (11), and (15) can be represented jointly with one compact matrix formula, as:

$$\begin{bmatrix} a^* + k_1^2 & 0 & 0 \\ 0 & a^* + k_2^2 & 0 \\ 0 & 0 & a^* + k_3^2 \end{bmatrix} \begin{bmatrix} U(x_2) \\ V(x_2) \\ W(x_2) \end{bmatrix} = k^2 \begin{bmatrix} U(x_2) \\ V(x_2) \\ W(x_2) \end{bmatrix}, \tag{19}$$

where

$$a^* = \frac{\mathrm{d}^2}{\mathrm{d}x_2^2}$$

Proceeding one step further, Eq. (19) can be written in a more abstract form as:

$$L\mathbf{f}(x_2) = \beta \mathbf{f}(x_2), \tag{20}$$

where $\mathbf{f}(x_2) = [U(x_2), V(x_2), W(x_2)]^{\mathrm{T}}$ is the eigenvector, and $\beta = k^2$ is the eigenvalue of the second order differential operator L, written in a matrix form, see Eq. (19).

The eigenvalue k^2 of the differential operator L is a complex number (Eq. (20)) in order to include the attenuation of the Love wave. Equation (20) together with the appropriate boundary conditions (Kiełczyński *et al.*, 2012), form a Direct Sturm-Liouville Problem for Love surface waves propagating in lossy waveguides (Kiełczyński, 2018). In general, solutions to the Direct Sturm-Liouville Problem form an infinite set of pairs $\{\lambda_n, \mathbf{f}_n(x_2)\}$, where $n = 0, 1, 2$ is the number of the wave mode. In this paper, we restricted our analysis to the propagation of the fundamental mode ($n = 0$) of the Love wave.

3.4. Boundary conditions

Boundary conditions at the interfaces between constituent layers shown in Fig. 1 are as follows:

1) At the free surface ($x_2 = -h_1$) of the Newtonian liquid layer, shear stress equals zero (Achenbach, 1973):

$$\tau_{23}^{(1)}\Big|_{x_2=-h_1} = 0. \tag{21}$$

2) On the surface ($x_2 = 0$), i.e. at the interface between the surface of the elastic layer and a viscous (Newtonian) liquid, continuity conditions for the mechanical displacement and shear stress should be provided, hence (Achenbach, 1973):

$$u_3^{(1)}\Big|_{x_2=0} = u_3^{(2)}\Big|_{x_2=0}, \tag{22}$$

$$\tau_{23}^{(1)}\Big|_{x_2=0} = c_{44}^{(1)} \frac{\partial u_3^{(1)}}{\partial x_2}\Bigg|_{x_2=0} = \tau_{23}^{(2)}\Big|_{x_2=0} = c_{44}^{(2)} \frac{\partial u_3^{(2)}}{\partial x_2}\Bigg|_{x_2=0}. \tag{23}$$

3) At the interface between the elastic surface layer and the elastic substrate ($x_2 = h_2$) the mechanical displacement u_3 and the shear stress τ_{23} must fulfill the conditions of continuity, i.e. (Achenbach, 1973):

$$u_3^{(2)}\Big|_{x_2=h_2} = u_3^{(3)}\Big|_{x_2=h_2}, \tag{24}$$

$$\tau_{23}^{(2)}\Big|_{x_2=h_2} = c_{44}^{(2)} \frac{\partial u_3^{(2)}}{\partial x_2}\Bigg|_{x_2=h_2} = \tau_{23}^{(3)}\Big|_{x_2=h_2} = c_{44}^{(3)} \frac{\partial u_3^{(3)}}{\partial x_2}\Bigg|_{x_2=h_2}. \tag{25}$$

4) The mechanical displacement $u_3^{(3)}$ of the surface Love wave, in the substrate ($x_2 \geq h_2$), must vanish for large distances from the interface ($x_2 = h_2$), i.e., $u_3^{(3)} \to 0$ for $x_2 \to \infty$.

4. Complex dispersion equation of the Love wave

After substitution of Eqs (8), (9), (12), (13), (16) and (17) into boundary conditions (Eqs (21)–(25)), the set of five linear and homogeneous equations for unknown coefficients C_1, C_2, C_3, C_4 and C_5 is obtained. For a nontrivial solution, the determinant of this set of linear algebraic equations must be equal to zero (necessary condition), namely:

$$\det \begin{bmatrix} b^* & c^* & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ c_{44}^{(1)} q_1 & 0 & -c_{44}^{(2)} q_2 & 0 & 0 \\ 0 & 0 & d^* & e^* & -f^* \\ 0 & 0 & \left(c_{44}^{(2)} q_2\right) e^* & \left(c_{44}^{(2)} q_2\right) d^* & \left(c_{44}^{(3)} b\right) f^* \end{bmatrix} = 0, \tag{26}$$

where

$$b^* = \cos(q_1 \cdot h_1), \qquad c^* = \sin(q_1 \cdot h_1),$$
$$d^* = \sin(q_2 \cdot h_2), \qquad e^* = \cos(q_2 \cdot h_2),$$
$$f^* = \exp(-b \cdot h_2).$$

Equation (26) leads to the following complex dispersion equation for Love waves propagating in the composite layered waveguide (see Fig. 1):

$$-\tan(q_1 \cdot h_1) \cdot \tan(q_2 \cdot h_2) \left\{ \frac{c_{44}^{(1)} \cdot q_1}{c_{44}^{(2)} \cdot q_2} \right\} - \tan(q_1 \cdot h_1) \left\{ \frac{c_{44}^{(1)} \cdot q_1}{c_{44}^{(3)} \cdot b} \right\} - \tan(q_2 \cdot h_2) \left\{ \frac{c_{44}^{(2)} \cdot q_2}{c_{44}^{(3)} \cdot b} \right\} + 1 = 0, \tag{27}$$

where the transverse wave number in Newtonian liquid $q_1 = \sqrt{\frac{\omega^2 \cdot \rho_1}{c_{44}^{(1)}} - k^2}$; the transverse wave number in elastic surface layer $q_2 = \sqrt{\frac{\omega^2 \cdot \rho_2}{c_{44}^{(2)}} - k^2}$; $k = k_0 + j\alpha$ is the complex wave number of the Love wave; $c_{44}^{(1)} = -j\omega\eta$ (in a Newtonian liquid); $b = \left(k^2 - k_3^2\right)^{1/2}$; $k_3 = \frac{\omega}{v_3}$ (in the substrate), h_1 and h_2 are the thicknesses of the Newtonian liquid layer and the elastic surface layer, respectively.

The complex dispersion equation (Eq. (27)) contains two real-valued unknowns, i.e. the real part k_0 of the complex wave number k of the Love wave and its attenuation α (i.e. the imaginary part of k).

The material parameters of the Newtonian liquid, elastic surface layer, and elastic substrate as well as the frequency f of the Love wave and the thicknesses h_1 and h_2 of the Newtonian liquid and the elastic surface layer are embedded in the dispersion Eq. (27) and are regarded as parameters of this equation.

The complex dispersion equation (Eq. (27)) can be written in a more abstract form as

$$\mathbf{F}\left(c_{44}^{(1)}, \rho_1, c_{44}^{(2)}, \rho_2, c_{44}^{(3)}, \rho_3, \eta, h_1, h_2, \omega; k_0, \alpha\right) = 0, \tag{28}$$

where bolded symbol $\mathbf{F}$ denotes that the equation is defined in the complex domain.

The complex dispersion equation (Eq. (29)) was subsequently split into its real and imaginary parts $\mathcal{R}e\mathbf{F}$ and $\mathcal{I}m\mathbf{F}$, which were further equated to zero, namely

$$\mathcal{R}e\mathbf{F}\left(c_{44}^{(1)}, \rho_1, c_{44}^{(2)}, \rho_2, c_{44}^{(3)}, \rho_3, \eta, h_1, h_2, \omega; k_0, \alpha\right) = 0, \tag{29}$$

$$\mathcal{I}m\mathbf{F}\left(c_{44}^{(1)}, \rho_1, c_{44}^{(2)}, \rho_2, c_{44}^{(3)}, \rho_3, \eta, h_1, h_2, \omega; k_0, \alpha\right) = 0. \tag{30}$$

Equations (29) and (30) constitute a system of two nonlinear transcendental algebraic equations for two unknowns k_0 and α. The parameters in Eqs (29), (30) are the following $c_{44}^{(1)}$, ρ_1, $c_{44}^{(2)}$, ρ_2, $c_{44}^{(3)}$, ρ_3, η, h_1, h_2, and ω. It is rather unrealistic to expect that any closed form solution for the system of two algebraic Eqs (29) and (30) would emerge. Therefore, the nonlinear system of two algebraic Eqs (29) and (30) have to be solved numerically.

The system of two nonlinear algebraic Eqs (29) and (30) was solved numerically using specialized procedures from the computer package Scilab. For the values of k_0, α found, the phase velocity of the Love surface wave was calculated from the following elementary formula $v_p = \omega / k_0$.

5. Results of numerical calculations

Numerical calculations were performed for Love waves propagating in the composite waveguide structure (see Fig. 1), consisting of (1) gold elastic surface layer and (2) Quartz elastic substrate. Top of the gold surface layer is loaded with a lossy Newtonian liquid. Numerical values for the material and geometrical parameters of the layered composite waveguide, used in the numerical calculations, are given below in Table 1.

For this set of material and geometric parameters of the Love wave waveguide and operating frequency $f = 1$ MHz, the maximum layer thickness ($h_{2\,\max}$) for monomode operation equals $h_{2\,\max} = 0.6$ mm (Auld, 1990). Therefore, only the fundamental Love wave mode is present in the considered waveguide structure. So that, in this paper we restricted our analysis to ultrasonic sensors that employ the fundamental mode of the Love wave.

5.1. Phase velocity and attenuation of the Love wave as a function of frequency for the Newtonian liquid of an infinite thickness

The dispersion curves of phase velocity and attenuation of the Love wave (i.e. the dependence of phase velocity v_p and attenuation α on frequency) are presented in Figs 2 and 3, respectively. Here, the Newtonian liquid, which loads the surface of the waveguide, is assumed to fill a semi-infinite half-space ($x_2 < 0$).

The impact of the finite thickness h_1 of a viscous Newtonian liquid layer on the phase velocity of the Love wave for various frequencies will be presented next in Figs 4–6.

5.2. Phase velocity and attenuation of the Love wave as a function of thickness h_1 of a Newtonian liquid layer

Plot of Love wave phase velocity v_p versus thickness h_1 of a Newtonian liquid layer, for $f = 0.2$ MHz is shown in Fig. 4.

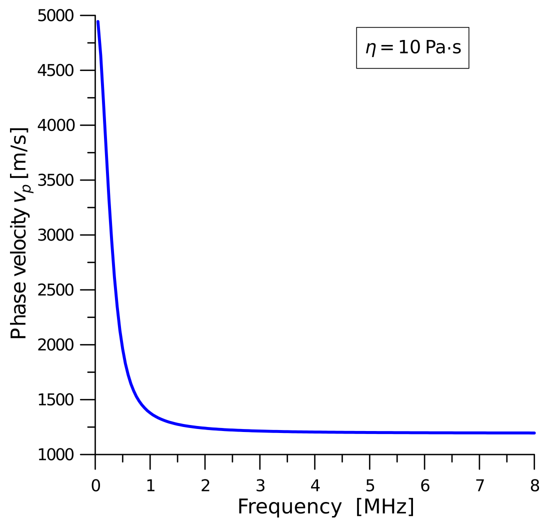

Fig. 2. Dispersion curve of the phase velocity of the Love wave propagating in the layered waveguide of Fig.1. Thickness of surface gold layer $h_2 = 0.1$ mm. Newtonian liquid of viscosity $\eta = 10$ Pa·s that loads the upper surface of the gold surface layer is a semi-infinite half space.

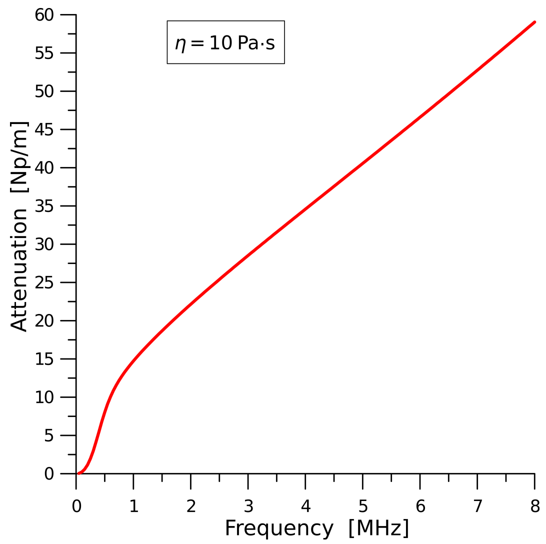

Fig. 3. Attenuation of the Love wave propagating in the layered waveguide of Fig. 1. Thickness of surface gold layer $h_2 = 0.1$ mm. Newtonian liquid of viscosity $\eta = 10$ Pa·s that loads the upper surface of the gold surface layer is a semi-infinite half space.

Figure 5 exhibits the plot of Love wave phase velocity v_p as a function of thickness h_1 of a Newtonian liquid layer, for $f = 0.5$ MHz.

Table 1. Material and geometrical parameters of the layered Love wave waveguide (Fig. 1), used in numerical calculations.

Material	Thickness [mm]	Density [kg/m^3]	Storage shear modulus [GPa]	SH wave velocity [m/s]	Viscosity [Pa·s]
Newtonian liquid	$h_1 = 0 - 1$	$\rho_1 = 1000$	0	NA	$\eta = 10$
Gold surface layer	$h_2 = 0.1$	$\rho_2 = 19\,300$	$c_{44}^{(2)} = 27.52$	$v_2 = 1194$	0
ST – cut Quartz substrate	semi-infinite	$\rho_3 = 2650$	$c_{44}^{(3)} = 67.85$	$v_3 = 5060$	0

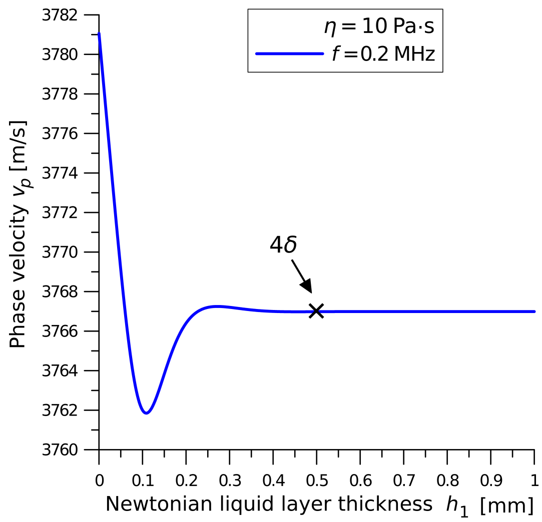

Fig. 4. Phase velocity v_p of the Love wave versus finite thickness (h_1) of the Newtonian liquid layer, that loads the waveguide surface. f = 0.2 MHz. δ is the penetration depth of the bulk SH wave into a Newtonian liquid.

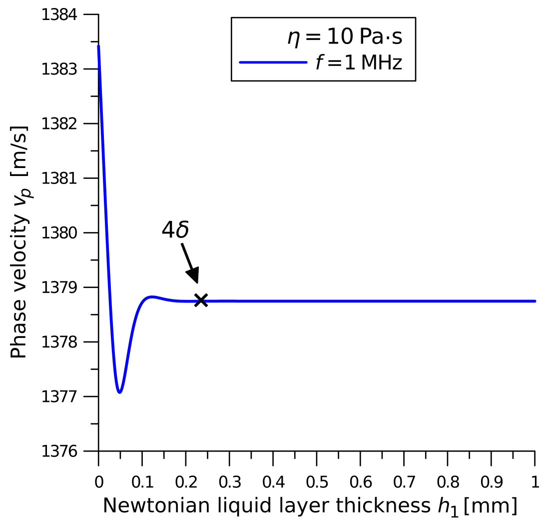

Fig. 6. Phase velocity v_p of the Love wave versus finite thickness (h_1) of the Newtonian liquid layer, that loads the waveguide surface. f = 1 MHz. δ is the penetration depth of the bulk SH wave into a Newtonian liquid.

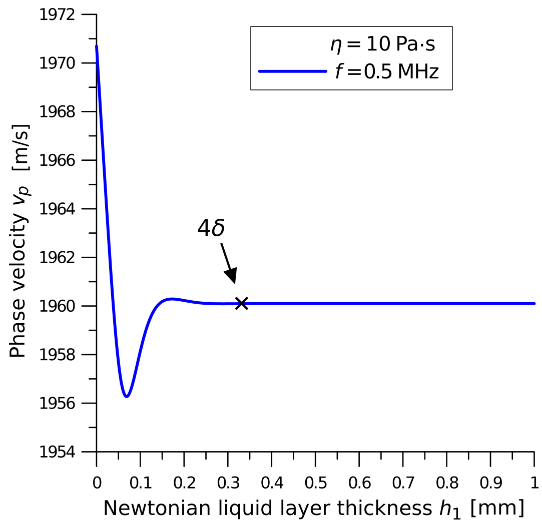

Fig. 5. Phase velocity v_p of the Love wave versus finite thickness (h_1) of the Newtonian liquid layer, that loads the waveguide surface. f = 0.5 MHz. δ is the penetration depth of the bulk SH wave into a Newtonian liquid.

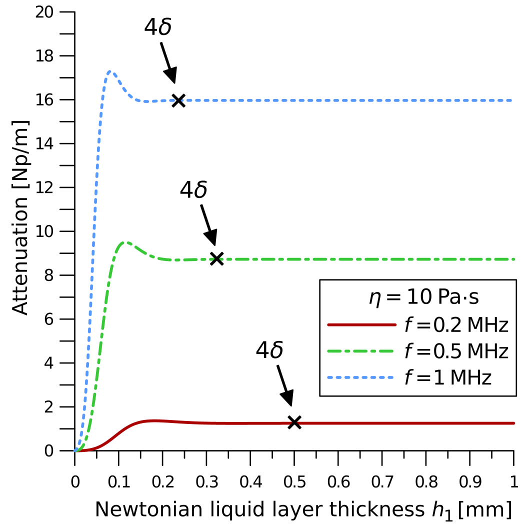

Fig. 7. Attenuation α of the Love wave as a function of Newtonian liquid layer thickness h_1. f = 0.2, 0.5 and 1 MHz. δ is the penetration depth of the bulk SH wave into a Newtonian liquid.

Curve in the Fig. 6 presents the Love wave phase velocity v_p as a function of thickness h_1 of a Newtonian liquid layer, for f = 1 MHz.

Attenuation α of the Love wave propagating in waveguides loaded with a Newtonian liquid as a function of h_1, for frequencies f = 0.2, 0.5 and 1 MHz, is shown in Fig. 7.

6. Discussion

If thickness h_1 of the loading Newtonian liquid is finite, we observe a strong dependence of the phase velocity v_p and attenuation α on the actual value of h_1 (see Figs 4–7). Initially, for the thickness h_1 of the loading Newtonian liquid growing from zero the phase velocity v_p drops monotonically and reaches a minimum for the thickness h_1 equaled approximatelly to the penetration depth δ ($h_1 \approx \delta$). Increasing further the thickness $h_1 > \delta$, we observe that the phase velocity v_p grows monotonically and attains a local maximum at $h_1 \approx 2\delta$. Next, (for $h_1 > 2\delta$) the phase velocity v_p slightly drops and enters a plateau for $h_1 > 4\delta$.

It means that if thickness of the loading Newtonian liquid $h_1 > 4\delta$, we can consider practically such a layer of Newtonian liquid as infinitive. This discovery is of crucial importance in design of Love wave sensors working in a liquid environment, such as biosensors, chemosensors etc.

Changes in the attenuation α, as a function of thickness h_1 (see Fig. 7), are strictly correlated with the corresponding changes in the phase velocity v_p as a function of h_1. In fact, the changes in α are mirror like with respect to changes in v_p, i.e. for growing phase velocity v_p the attenuation is a decreasing function of h_1. If v_p reaches a minimum ($h_1 \approx \delta$), the attenuation attains a maximum etc.

In general, a layer of a Newtonian liquid of any thickness, different from zero ($h_1 > 0$), will decrease phase velocity v_p of the Love surface wave (see Figs 4–6). Assuming that thickness of the Newtonian liquid is larger than $h_1 > 4\delta$ (infinite-like thickness), the phase velocity v_p changes from 3781 m/s (for $h_1 = 0$) to 3765 m/s (for $f = 0.2$ MHz), from 1971 m/s (for $h_1 = 0$) to 1959 m/s (for $f = 0.5$ MHz) and from 1383 m/s (for $h_1 = 0$) to 1379 m/s (for $f = 1$ MHz).

The penetration depth of the Love surface wave into a loading Newtonian liquid requires some explanations. It is a known fact that for bulk SH waves the penetration depth δ into a Newtonian liquid can be expressed by the following exact formula (Ballantine *et al.*, 1997):

$$\delta = (2\eta/\omega\rho)^{1/2}, \tag{31}$$

where η is the viscosity of the liquid, ρ is the density of the liquid, and ω is the wave angular frequency.

Although the penetration depth δ, given by Eq. (31), is strictly valid only for bulk SH waves, it approximately equals (with a high accuracy) the penetration depth of the Love surface wave, for the Newtonian liquid of viscosity $\eta = 10$ Pa·s and density $\rho = 10^3$ kg/m^3 at frequencies 0.2–1 MHz. Therefore, in this paper we use Eq. (31) to evaluate the penetration depth of the Love surface wave, into the loading Newtonian liquid with parameters given in Table 1.

For the Newtonian liquid of viscosity $\eta = 10$ Pa·s and density $\rho = 10^3$ kg/m^3, the penetration depth δ as a function of frequency f of the Love wave equals, respectively, to: $\delta = 0.126$ mm, for $f = 0.2$ MHz; $\delta = 0.08$ mm, for $f = 0.5$ MHz; and $\delta = 0.056$ mm, for $f = 1$ MHz.

7. Conclusions

The authors established the complex dispersion equation (Eq. (27)) for Love waves propagating in the layered waveguide structure from Fig. 1. Using this complex dispersion equation, the dispersion curves of phase velocity and attenuation have been evaluated.

From these dispersion curves for phase velocity and attenuation (see Figs 4–7) of Love surface waves, propagating in waveguides loaded with a Newtonian liquid of a finite thickness h_1, we can draw the following conclusions:

1) The change of the phase velocity v_p of the Love wave as a function of thickness h_1 of the loading Newtonian liquid has character of a damped sinusoid, for small initial values of the thickness h_1, e.g., for $f = 1$ MHz ($h_1 < 0.3$ mm), see Fig. 6. After exceeding the thickness $h_1 \approx 0.3$ mm (for $f = 1$ MHz) we can treat the Newtonian liquid as semi-infinite.
2) In general, the dispersion curves of the phase velocity v_p as a function of thickness h_1, for frequencies 0.2, 0.5, and 1 MHz, see Figs 4–6, show that the limiting thickness beyond which a Newtonian liquid can be treated as a semi-infinite can be safely assumed as 4δ (four penetration depths into the liquid for a given frequency).
3) The variation of the Love wave attenuation α as a function of thickness h_1 has also character of a damped sinusoid in the range of small values of thickness h_1, see Fig. 7.
4) Figure 7 shows that, similarly as in the case of the phase velocity v_p dependence on the thickness h_1, Newtonian liquid layer having a thickness of approximately greater than four penetration depth 4δ can be safely regarded as a semi-infinite half space.

The results presented in this work have not been yet published in the scientific literature. We believe that this work will allow for a more accurate analysis of physical phenomena occurring in ultrasonic Love wave sensors. This will enable more precise design and optimization of the Love wave ultrasonic sensors (e.g. viscosity sensors, biosensors and chemosensors).

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