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APPLICATION OF THE NARMAX METHOD TO THE MODELLING OF THE NONLINEARITY OF DYNAMIC LOUDSPEAKERS

A.B. DOBRUCKI and P. PRUCHNICKI

Institute of Telecommunications and Acoustics,
Wrocław University of Technology
(50-370 Wrocław, Wybrzeże S. Wyspiańskiego 27, Poland)
e-mail: ado@zakus.ita.pwr.wroc.pl, misio@zakus.ita.pwr.wroc.pl

The application of the NARMAX method to the modelling of the nonlinearity of dynamic loudspeakers is described. The principle of creating a polynomial representation of a model, the problems stemming from a too large number of model coefficients and the method of optimizing the model are presented. The method was tested on data from actual loudspeaker measurements. Different models are compared as regards their accuracy depending on the modelling parameters. Finally, the model characteristics are compared with the results of loudspeaker measurements performed by classical methods.

1. Introduction

Loudspeaker nonlinearity can be modelled by various methods such as Volterra series [7, 12], nonlinear analogous equivalent circuits [9], nonlinear differential equations [6] and so on. One of the methods is NARMAX (**N**on-linear **A**uto**R**egressive **M**oving **A**verage with e**X**ogenous input). The NARMAX model was proposed by LEONTARITIS and BILLINGS in 1985 [11, 12]. In this model, the output signal values are computed using both the input signal values and the previous output signal values. This greatly reduces the number of coefficients.

The NARMAX model is then analogous to IIR (Infinite Impulse Response) digital filters similarly as the Volterra series model is analogous to FIR (Finite Impulse Response) digital filters. FIR-filters use only input signal samples and require a large number of coefficients. IIR-filters use both input and output signal samples and require a much smaller number of coefficients. The above terminology is used in this paper.

The polynomial NARMAX model for the dynamic loudspeaker is described in the paper. It has been proved that the direct model can be unstable. In order to stabilize the model, the optimization procedure is necessary. The optimization causes also significant reduction of the number of coefficients. The modeling of an actual loudspeaker has been done, and the results of the modeling and measurements are compared.

2. Polynomial representation of the NARMAX model

The most general NARMAX model of a system with one input and one output can be expressed by the following equation:

$$y(t) = F[y(t-1), ..., y(t-n_y),\ x(t-d), ..., x(t-d-n_x),\ e(t-1), ..., e(t-n_e)] + e(t), \qquad (1)$$

where $F[\cdot]$ — an unknown nonlinear function, t — the discrete time, $x(t)$ — the excitation, $y(t)$ — the system response, $e(t)$ — the prediction error, n_x — the order of the input signal, n_y — the order of the output signal, n_e — the order of the noise, d — the delay of the system.

If it is assumed that the system does not produce any noise, a simplified form of the NARMAX model can be developed. The latter can be described by the following general equation [1, 4]:

$$y(t) = F\left[y(t-1), \ldots, y(t-n_y),\ x(t-d), \ldots, x(t-d-n_x)\right] + e(t). \qquad (2)$$

Polynomial functions are most commonly applied as the F functions, although other functions, e.g. rational or radial ones, can also be used [2, 4]. The polynomial representation of the NARMAX model is as follows:

$$y(t) = \sum_{i_1=0}^{n} \theta_{i_1} u_{i_1}(t) + \sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} \theta_{i_1 i_2} u_{i_1}(t) u_{i_2}(t) + \sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} \sum_{i_3=i_2}^{n} \theta_{i_1 i_2 i_3} u_{i_1}(t) u_{i_2}(t) u_{i_3}(t) + \ldots + e(t), \qquad (3)$$

where $n = n_y + n_x$, $u_1(t) = y(t-1)$, $u_2(t) = y(t-2)$, ..., $u_{ny}(t) = y(t-n_y)$, $u_{ny+1}(t) = x(t-d)$, ..., $u_n(t) = x(t-d-n_x)$, θ — model coefficients.

Equation (3) can be written as:

$$y(t) = \sum_{m=1}^{M} \theta_m p_m(t) + e(t), \qquad (4)$$

where M — the number of polynomial coefficients, $p_m(t)$ — the monomials of elements $u_i(t)$ of degree l at the most.

For example, for $n_y = n_x = l = 2$ there are $M = 20$ polynomials and they are as follows:

$$\begin{aligned}
p_1(t) &= y(t-1), & p_2(t) &= y(t-2),\\
p_3(t) &= x(t-d), & p_4(t) &= x(t-d-1),\\
p_5(t) &= x(t-d-2), & p_6(t) &= y^2(t-1),\\
p_7(t) &= y^2(t-2), & p_8(t) &= x^2(t-d),\\
p_9(t) &= x^2(t-d-1), & p_{10}(t) &= x^2(t-d-2),\\
p_{11}(t) &= y(t-1)\cdot y(t-2), & p_{12}(t) &= y(t-1)\cdot x(t-d),\\
p_{13}(t) &= y(t-1)\cdot x(t-d-1), & p_{14}(t) &= y(t-1)\cdot x(t-d-2),\\
p_{15}(t) &= y(t-2)\cdot x(t-d), & p_{16}(t) &= y(t-2)\cdot x(t-d-1),\\
p_{17}(t) &= y(t-2)\cdot x(t-d-2), & p_{18}(t) &= x(t-d)\cdot x(t-d-1),\\
p_{19}(t) &= x(t-d)\cdot x(t-d-2), & p_{20}(t) &= x(t-d-1)\cdot x(t-d-2).
\end{aligned}$$

If we have N input and output signal samples obtained from measurements, from Eq. (4) we can develop a system of equations which can be expressed in following matrix form [4, 5]:

$$\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} p_1(1) & p_2(1) & \dots & p_M(1) \\ p_1(2) & p_2(2) & \dots & p_M(2) \\ \dots & \dots & \dots & \dots \\ p_1(N) & p_2(N) & \dots & p_M(N) \end{bmatrix} \cdot \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_M \end{bmatrix} + \begin{bmatrix} e(1) \\ e(2) \\ \vdots \\ e(N) \end{bmatrix} \tag{5}$$

and in this simpler form:

$$\mathbf{Y} = \mathbf{P}\boldsymbol{\theta} + \mathbf{e}. \tag{6}$$

System (6) is a linear equation system since the terms of regression matrix $\mathbf{P}$ are numbers calculated from the measured data. The model is identified by the solution of system (6), where coefficients θ_1 ... θ_m are unknown. Prediction error vector $\mathbf{e}$ is assumed to be equal to 0.

There are various methods of solving a linear equation system, e.g. Gauss elimination or iterative methods [8]. System (6) is often ill-conditioned and therefore matrix $\mathbf{P}$ should be orthogonalized [1, 5] using, for example, the Gram-Schmidt method, the Givens rotations or the Householder transformation [8]. In this paper the classical Gram-Schmidt (CGS) orthogonalization is applied since it can be easily implemented in numerical computations.

The orthogonalization algorithm is based on the decomposition of the prediction matrix into two matrices [1, 5].

$$\mathbf{P} = \mathbf{W}\,\mathbf{A}, \tag{7}$$

where $\mathbf{W}$ is columnwise orthonormal, i.e. $\mathbf{W}^T\mathbf{W} = \mathbf{I}$, $\mathbf{I}$ is a unit matrix, $\mathbf{A}$ is a triangular upper matrix.

After orthogonalization, vector $\mathbf{g}$ is determined.

$$\mathbf{g} = \mathbf{W}^T\mathbf{Y}. \tag{8}$$

Then taking advantage of the fact that matrix $\mathbf{A}$ is triangular, reverse substitution is applied to determine coefficients θ:

$$\mathbf{A}\boldsymbol{\theta} = \mathbf{g}. \tag{9}$$

The main drawback of polynomial representation is that a very large number of parameters must be determined. The number of coefficients (M) in the polynomial which describes the model depends on the lag of the input and output signals and that of the noise and on the particular order of the polynomial (order of nonlinearity — l). The number can be determined from this recurrence formula [2, 14]:

$$M = \sum_{i=1}^{l} n_i\,, \qquad n_i = [n_{i-1}(n_y + n_x + n_e + i)]/i, \quad n_0 = 1. \tag{10}$$

For example for $n_y = n_x = n_e = 10$ and $l = 3$ the number of coefficients is as high as $M = 5983$ and the number of terms in matrix $\mathbf{P}$ is equal to $M \cdot N$ where $N \geq M$ (most often $N > M$).

In order to compare various models, the following measure of accuracy is assumed:

$$\varepsilon = \frac{\|e\|^2}{\|Y_r\|^2} = \frac{\sum (y_r - y_m)^2}{\sum y_r^2} \cdot 100\%, \tag{11}$$

where $\mathbf{Y}_r$ — the vector of the response of an actual loudspeaker — $\mathbf{Y}_r = \mathbf{P}\theta + \mathbf{e}$, $\mathbf{Y}_m$ — the vector of the response of the model — $\mathbf{Y}_m = \mathbf{P}\theta$, $\|a\| = \sqrt{\sum_i a_i^2}$ — Euclid's norm of the vector, ε is a ratio of the prediction error energy to the energy of the response.

3. Optimization of the model

Since the number of coefficients to be calculated is very large (due to the fact that the structure of the nonlinearity of the modelled actual system is unknown), the usefulness of such a model is rather small. There are also difficulties in the correct interpretation of the model. In addition, the unoptimized model is unstable in most cases.

In order to optimize the model, it is necessary to reject the insignificant coefficients, i.e. to identify the structure of the system.

The optimizing procedure has been built into the orthogonalization algorithm (CGS) for regression matrix $\mathbf{P}$. It is based on the choice of subset M_s ($M_s < M$) of columns from all possible columns M of matrix $\mathbf{P}$ (Fig. 1). This yields new regression matrix $\mathbf{P}_s$ with a lower number of coefficients. The columns of $\mathbf{P}_s$ are selected using this error reduction ratio [1, 5]:

$$[\text{err}]_i = \frac{g_i^2}{\langle y, y \rangle}, \tag{12}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product, that is:

$$\langle y, y \rangle = \sum_{k=1}^{N} y_k^2(t) \tag{13}$$

and g_i is i-th term of vector $\mathbf{g}$.

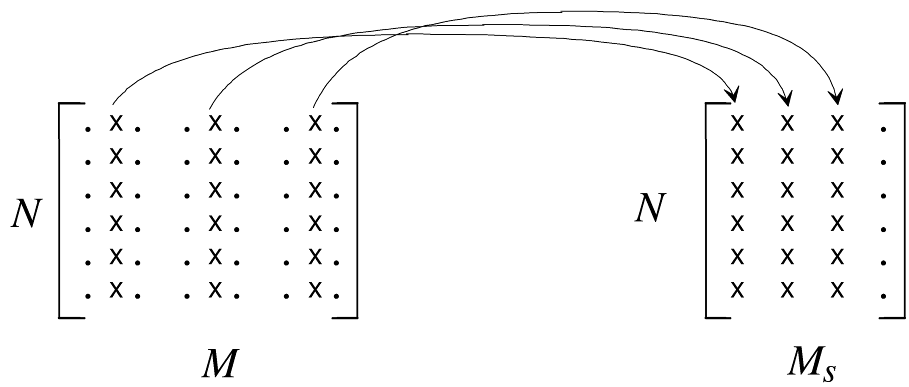

Fig. 1. Selection of most significant columns of regression matrix.

The value of $[\mathrm{err}]_i$ represents a decrease in the prediction error energy for coefficient θ_i expressed by column $\mathbf{p}_i$.

The optimizing procedure performs the following functions in every step of orthogonalization:

— the computation of error reduction ratio $[\mathrm{err}]_i$ for every coefficient,

— the choice of a coefficient with the maximum value of $[\mathrm{err}]_i$.

Now the size of matrix $\mathbf{P}_s$, i.e. the number of coefficients θ_i, remains to be determined. The accuracy of the model: ρ $(0 < \rho \leq 1)$ is often assumed as the optimization-end criterion [14]. The coefficients are selected as long as Eq. (14) is not fulfilled.

$$1 - \sum_{i=1}^{M_s} [\mathrm{err}]_i < \rho. \tag{14}$$

This criterion has a disadvantage. When a high model accuracy (a low value of ρ) is assumed, too many coefficients (often all of them $(M_s = M)$) may be taken into account. Akaike's information criterion (15) gives better results [1, 5, 14].

$$\mathrm{AIC}(\phi) = N \log \sigma_e^2 + M_s \phi, \tag{15}$$

where $\sigma_e^2 = \dfrac{1}{N} \sum\limits_{i=1}^{N} e_i^2$ — prediction error variance.

This criterion represents a compromise between the accuracy of the model (σ_e^2) and its compliance (M_s). The NARMAX model structure is usually defined by means of $\phi = 4$ (AIC(4)) [13]. The formation of $\mathbf{P}_s$ is stopped when AIC(4) reaches the minimal value.

4. High-order model

Because of the long loudspeaker impulse response, a satisfactory accuracy can be obtained only if the order of the model is sufficiently high but this entails a large number of coefficients. For example, to obtain a NARMAX model of the 30th order, a matrix consisting of about 50 000 columns must be orthogonalized. It is practically impossible to handle this amount of data — prediction matrix $\mathbf{P}$ would use about 20 GB of memory. Therefore a way had to be found to overcome this problem.

The fact that the number of coefficients can be reduced many times through the optimization procedure was exploited. The model is built in steps which are graphically represented in Fig. 2. First a low-order model with M_i: (300 – 500) coefficients is created and optimized. As a result, a model with maximally a few dozen coefficients is obtained. Then the model is supplemented with the next M_i coefficients due to its increased order. After the next optimization, again a few dozen coefficients (not necessarily the same as in the first step) are obtained. This procedure is repeated many times until all the coefficients associated with the assumed order of the model have been analyzed.

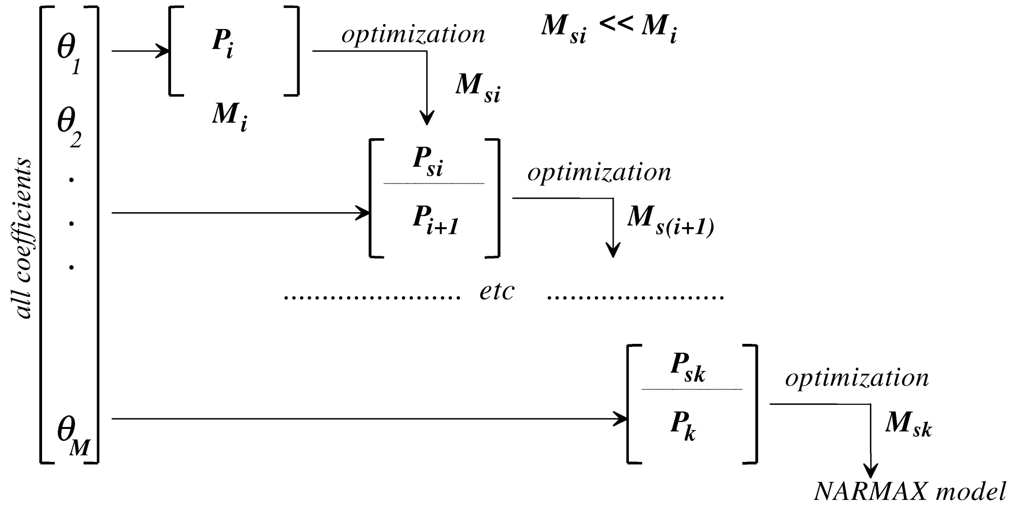

Fig. 2. Steps in creation of high-order NARMAX model.

5. Comparison of the results of measurements and modelling

5.1. Measurements

In order to obtain data (excitation $x(t)$ and response $y(t)$ values) for the creation of the model, measurements of a low-frequency loudspeaker were performed in the anechoic chamber of the Institute of Telecommunications and Acoustics. The loudspeaker (Tonsil GDN 20/35/1) was set in a closed box and digitally generated noise with uniform probability distribution and an amplitude of 15 V RMS (28 W) (2/3 of the loudspeaker's nominal power) was used as the input signal [3]. The loudspeaker response was recorded via a microphone and converted to a digital domain. In order to eliminate random noise, the response was averaged 100 times. The measurement setup is shown in Fig. 3.

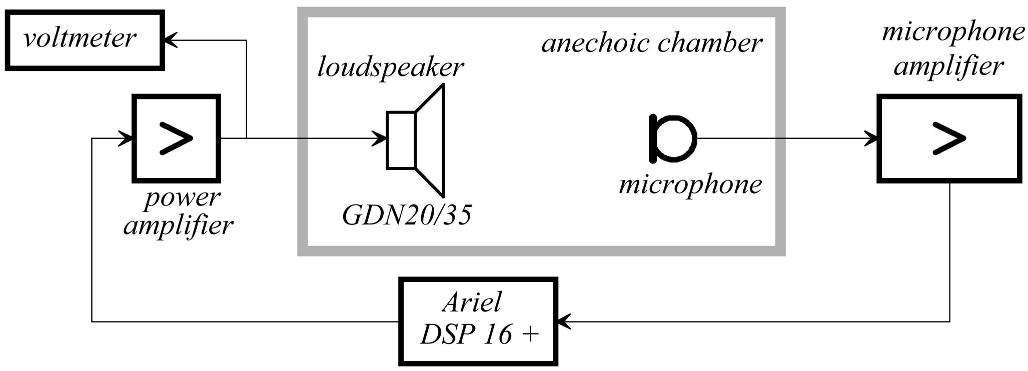

Fig. 3. Measurement setup.

5.2. Unoptimized NARMAX model

A linear FIR-type model of the 50-th order was investigated first. The impulse response of the model was compared with that of the loudspeaker — see Fig. 4. The accuracy was quite good, particularly in the initial part of the response.

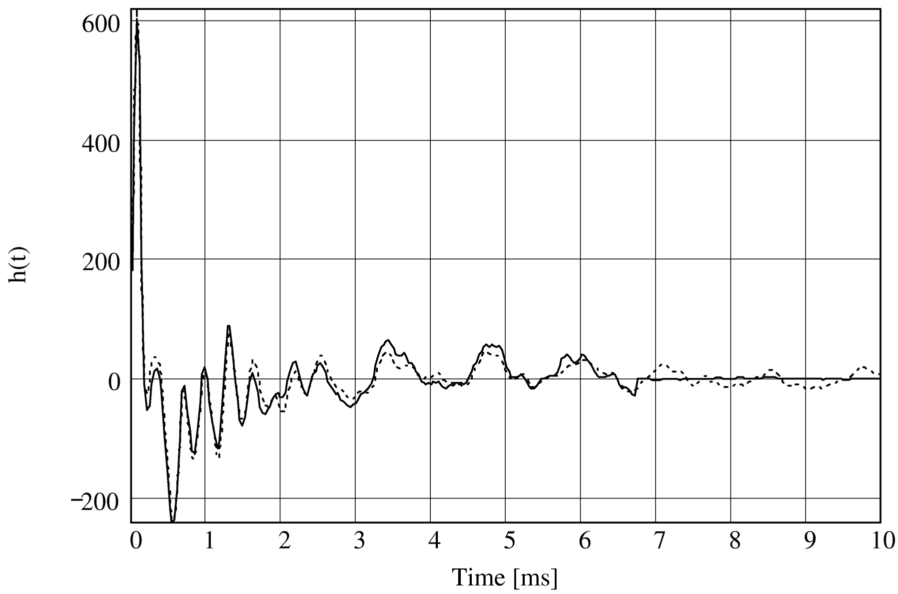

Fig. 4. Loudspeaker impulse response obtained from measurement (dashed line) and from model (solid line).

Then two IIR-type linear models of the 20-th and 50-th order were identified. Finally, FIR-type nonlinear models of the 8-th and 12-th order and IIR-type (i.e. NARMAX) models of the 4-th and 6-th order were studied. In all the nonlinear models, the order of nonlinearity was limited to $l = 3$.

The IIR-type models (which use also delayed system response samples) proved to be unstable. The number of coefficients and error ε for all the considered models are given in Table 1.

Table 1. Comparison of different loudspeaker models.

Model (structure)	ε [%]	No of coeff.
$l = 1$, $n_x = 50$, $n_y = 0$, (FIR)	8.65	51
$l = 1$, $n_x = 200$, $n_y = 0$, (FIR)	5.18	201
$l = 1$, $n_x = 20$, $n_y = 20$, (IIR)	–	41
$l = 1$, $n_x = 50$, $n_y = 50$, (IIR)	–	101
$l = 3$, $n_x = 8$, $n_y = 0$, (nonlinear FIR)	53.7	165
$l = 3$, $n_x = 12$, $n_y = 0$, (nonlinear FIR)	111.6	560
$l = 3$, $n_x = 4$, $n_y = 4$, (nonlinear IIR)	–	165
$l = 3$, $n_x = 6$, $n_y = 6$, (nonlinear IIR)	–	680

5.3. Optimized model

The results of the loudspeaker measurements described in Subsec. 5.1 were used to check the optimization procedure. Models with different structure (identical as in Subsec. 5.2) were considered. The optimization results are presented in Table 2.

Table 2. Comparison of various loudspeaker models (after optimization).

Model (structure)	ε [%]	No of coeff.
$l=1$, $n_x=50$, $n_y=0$, (FIR)	8.43	30
$l=1$, $n_x=200$, $n_y=0$, (FIR)	6.44	52
$l=1$, $n_x=20$, $n_y=20$, (IIR)	10.57	26
$l=1$, $n_x=50$, $n_y=50$, (IIR)	6.12	43
$l=3$, $n_x=8$, $n_y=0$, (nonlinear FIR)	34.03	6
$l=3$, $n_x=12$, $n_y=0$, (nonlinear FIR)	35.19	8
$l=3$, $n_x=4$, $n_y=4$, (nonlinear IIR)	37.61	9
$l=3$, $n_x=6$, $n_y=6$, (nonlinear IIR)	36.54	10

The characteristic feature of all the optimized models is their stability. For a similar or higher accuracy than that of the unoptimized models they require a much lower number of coefficients.

An illustrative impulse response of the 4-th order NARMAX model is shown in Fig. 5. The response is very short, which means that the order of the model is too low.

A comparison of the plots for the loudspeaker and the model excited by the same signal (Fig. 6) shows that the model has a tendency to reduce the maximum amplitude values.

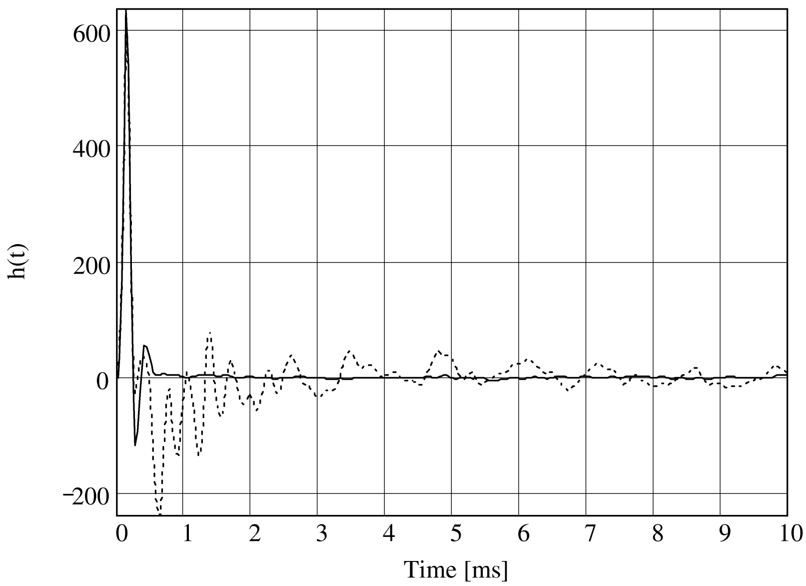

Fig. 5. Loudspeaker impulse response obtained from measurement (dashed line) and from 4-th order, 10-coefficient NARMAX model (solid line).

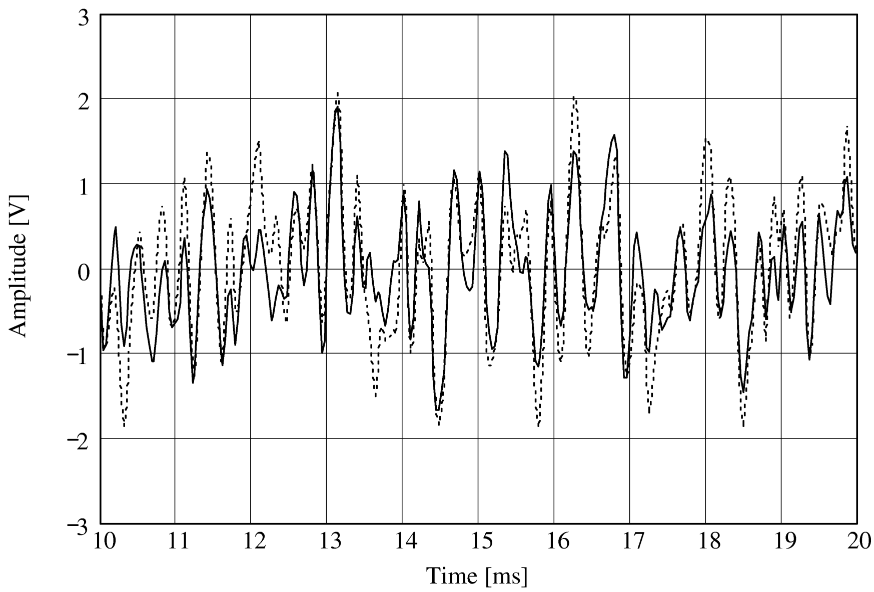

Fig. 6. Signal at loudspeaker output (dashed line) and from 4-th order, 10-coefficient NARMAX model (solid line).

The four coefficients which if included in the model ensure the highest error reduction values [err] are given in Table 3. They have the most decisive effect on the accuracy of the model and are selected as the first ones by the optimization procedure. The coefficients are linear since the loudspeaker nonlinearities were slight.

Table 3. Coefficients ensuring highest [err] values.

Model structure	1		2		3		4	
	term	[err]	term	[err]	term	[err]	term	[err]
FIR and NFIR	$x(t-2)$	0.20	$x(t-3)$	0.22	$x(t-1)$	0.14	$x(t-4)$	0.05
IIR and NIIR	$y(t-1)$	0.82	$y(t-2)$	0.10	$y(t-3)$	0.04	$x(t-1)$	0.01

To gain a picture of the relationship between model accuracy and the number of coefficients, a group of models was built. All the models were developed for the same signal and parameters:

— the order of nonlinearity — $l = 3$,

— the order of the model — $n_x = n_y = 16$,

— the criterion for the choice of coefficients — error reduction ratio [err],

— the criterion for ending model development — ρ.

Only the value of ρ was changed to obtain models with different numbers of coefficients. Also modelling for the termination criterion based on AIC(4) was performed to find out when the selection of coefficients will end.

The relationship between the modelling error and the number of coefficients is illustrated in Table 4 and Fig. 7.

Table 4. Relationship between modelling error and number of coefficients.

M_s (No of coeff.)	ε [%]	ρ [%]
7	44.56	2.00
9	24.14	1.00
11	15.22	0.70
13	13.41	0.60
15	14.53	0.50
16	17.26	0.45
18	17.39	0.40
20	14.33	0.37
21	17.63	0.35
24	15.19	0.33
27	13.51	0.30
28	13.19	0.28
29	11.27	AIC(4)
32	12.60	0.26
34	12.77	0.25

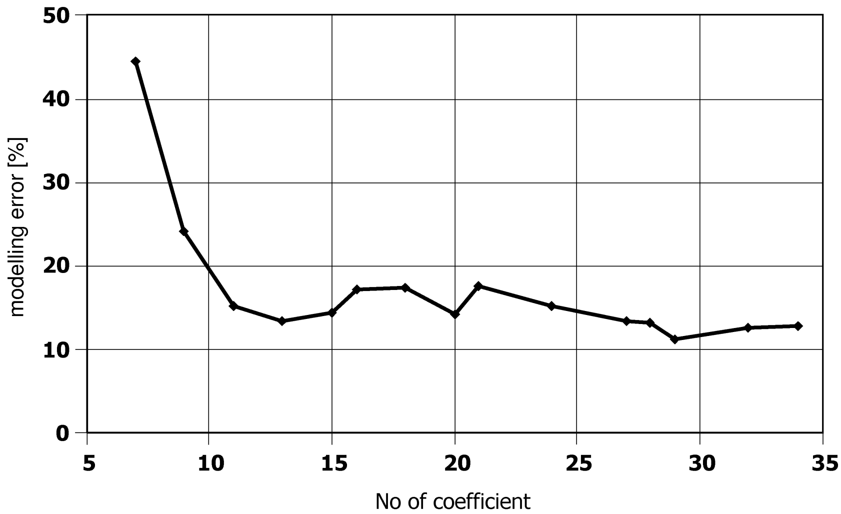

Fig. 7. Relationship between modelling error and number of coefficients for NARMAX model.

5.4. High-order model

A NARMAX model was built according to the algorithm described in Sec. 4 for the following modelling parameters:

— the order of nonlinearity — $l = 3$,

— the order of the model — $n_x = n_y = 33$,

— the number of rows in matrix **P** — $N = 700$,
— the number of coefficients determined in one step — $M_i = 400$,
— the number of samples for model testing — $N_f = 8192$,
— the criterion for the selection of coefficients — error reduction ratio [err],
— the criterion for ending the creation of the model — AIC(4).

The modelling resulted in a 64-coefficient NARMAX model characterized by error $\varepsilon = 13.9\%$.

The response of the model and that of the actual loudspeaker to the same excitation are shown in Fig. 8; the impulse responses and the frequency characteristics are shown respectively in Figs. 9 and 10.

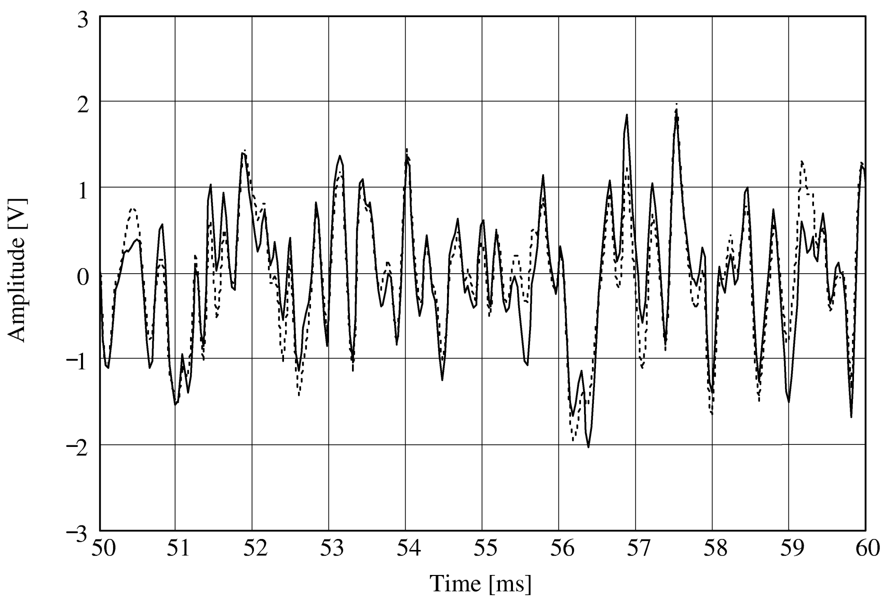

Fig. 8. Model response (solid line) and actual loudspeaker response (dashed line).

By comparing the impulse responses and the frequency characteristics we can assess only the linear properties of the model. To check how the model copes with nonlinearities, THD (a coefficient commonly used for assessing nonlinear distortions) was employed. To obtain the data needed for the calculation of THD, sinusoidal excitations with different frequencies were fed at the loudspeakers input and a spectral analysis of the loudspeaker responses was carried out, yielding the levels of the particular harmonics. The same excitation signals were fed at the input of the model and the latter's response was analyzed. Spectra of the response to the 300 Hz sinusoidal signal excitation are shown in Fig. 11. To see them better, the two spectra are shifted slightly relative to each other on the frequency axis. No components higher than the third harmonic occur in the model response spectrum (the left spectral lines) — due to the fact that the model nonlinearity was limited to the 3rd order.

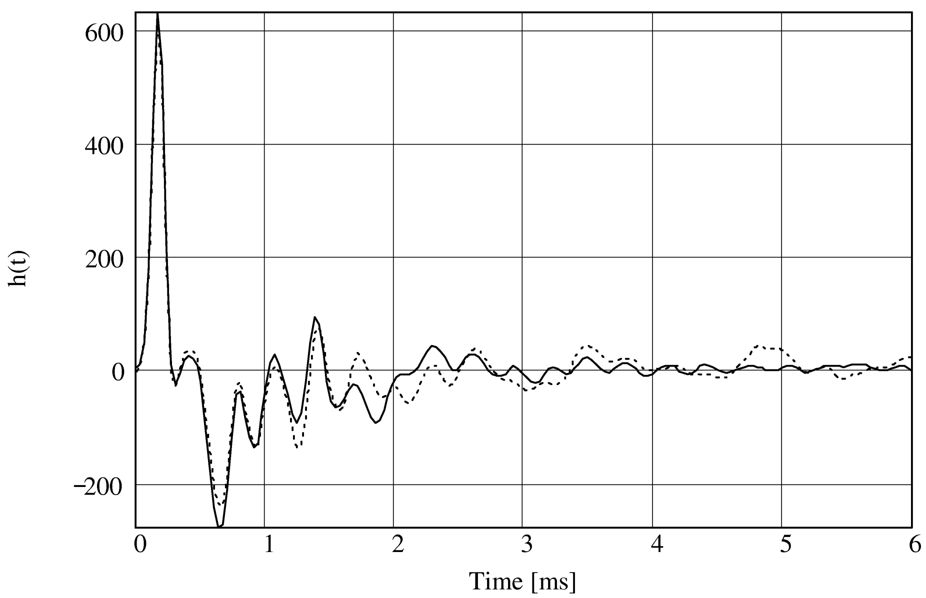

Fig. 9. Model impulse response (solid line) and actual loudspeaker impulse response (dashed line).

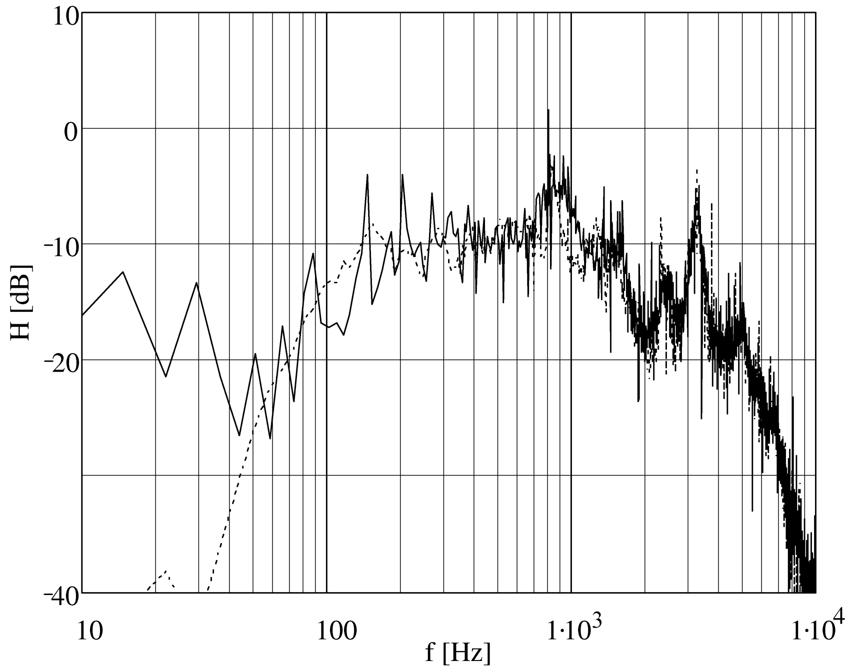

Fig. 10. Model frequency characteristic (solid line) and actual loudspeaker frequency characteristic (dashed line).

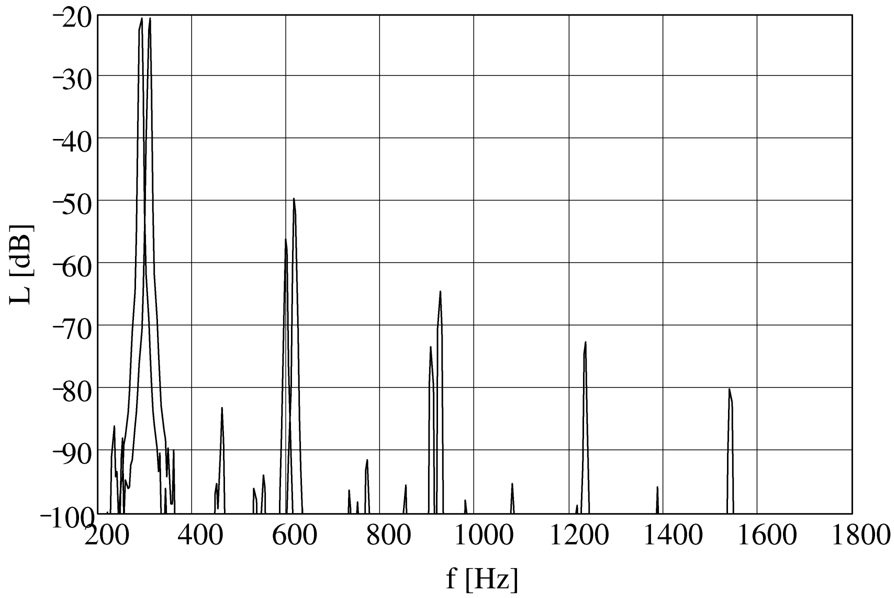

Fig. 11. Spectra of loudspeaker response (right spectral lines) and model response (left spectral lines) to 300 Hz sinusoidal signal excitation.

The THD (dB) for the first three harmonics was calculated from the following formula:

$$L_{\mathrm{THD}} = 10\log(10^{L_2/10} + 10^{L_3/10}) - L_1 \quad [\mathrm{dB}]. \tag{16}$$

The results are given in Table 5.

Table 5. THD for actual loudspeaker and model responses.

Frequency [Hz]	Loudspeaker THD [dB]	Model THD [dB]
100	−21.9	−39.1
150	−25.9	−48.2
200	−25.5	−33.2
300	−28.9	−35.1
500	−30.7	−35.6
700	−38.5	−44.0
1000	−37.8	−49.3
2000	−33.1	−51.8
4000	−49.5	−73.6

6. Conclusions

The simulations and the measurements have shown that to model a dynamic loudspeaker correctly it is necessary to use a high-order NARMAX model. A polynomial representation of such a model requires a very large number of coefficients. Besides the obvious computational and interpretational problems associated with operations on such a large set of data, it is also difficult to obtain stability. All the unoptimized NARMAX models proved to be unstable (Table 1).

Therefore an optimization procedure was applied and as a result the number of coefficients was reduced considerably whereby the stability of the model improved (Table 2).

It follows from Table 3 that the chosen criterion (based on AIC(4)) for ending the selection of model coefficients ensured the highest accuracy of the model in the analyzed range of numbers of coefficients.

An analysis of the higher-order model showed a close similarity between the model linear characteristics and the actual loudspeaker linear characteristics. Some differences can be observed between the frequency characteristics — the model one is more jagged and irregular.

The model response has smaller linear distortions owing to the fact that no terms with higher orders of nonlinearity occur in the model: their presence would result in the appearance of higher harmonics and increase the level of the second and third harmonic.

Acknowledgments

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