ARCHIVES OF ACOUSTICS
22, 1, 51–58 (1997)

REFRACTION – THE SIMPLEST CASE

R. MAKAREWICZ

Kyushu Institute of Design
9-1 Shiobaru 4-chome
(Minami-ku, Fukuoka, 815 Japan)

R. GOŁĘBIEWSKI

Institute of Acoustics, A. Mickiewicz University
(60-769 Poznań, ul. Matejki 48)

At longer ranges of outdoor sound propagation, refraction due to temperature and wind variations results in ray paths that are curved. Under the assumptions of the linear effective sound speed and nearly horizontal propagation, the ray path in the form of parabola is used. The shape of the ray and the position of the shadow zone, in the presence of a negative gradient, is studied. In the converse case of a positive gradient, the analysis of the additional reflected rays is performed. This is the most simple case of the theory of refraction.

1. Introduction

Sound propagation outdoors involves a number of wave phenomena, among them, refraction. It is assumed that the atmospheric surface layer is stratified, therefore the sound speed, $\widetilde{c}$, and the speed of wind, V (blowing along the x-axis), are functions of height z. For simplicity we ignore the crosswind, so that rays from the source stay within a vertical plane, $\Phi = \text{const}$ (Fig. 1). Thus, a ray undergoes refraction as if it were moving in the atmosphere with no wind, but with an effective sound speed [10],

$$c(z) = \widetilde{c} + V(z)\cdot\cos\Phi. \tag{1.1}$$

In other words, the effective sound speed, c, is the sum of the local sound speed and the component of the wind speed in the direction of propagation. Within the scope of geometrical acoustics, many functions have been suggested to model the real profile of $c(z)$ [4, 6, 8, 13]. Among them, the linear function,

$$c(z) = c(0)\cdot(1 + az), \tag{1.2}$$

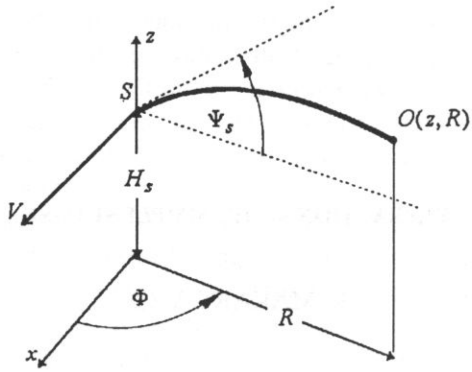

Fig. 1. Geometry of source, S, and receive, O. The wind is in the direction of the x-axis.

with $c(0)$ expressing the sound speed on the ground, outlines the most salient feature of the effective sound speed: either it increases ($a > 0$) or decreases ($a < 0$) with height. Note that the above equation is theoretical assumption that yields the average outcome of both sound speed and wind speed variations. Therefore, the coefficient a is the equivalent sound speed gradient. The results obtained by EMBLETON *et al.* [2], HIDAKA *et al.* [5], RASMUSSEN [11], and most recently, by L'ESPERANCE *et al.* [7], show that the linear profile explains some field data quite satisfactorily.

For the source on the ground (Fig. 1 with $H_s = 0$), the linear function $c(z)$ yields the ray path in the form of a circle [3, 9, 10, 12]. It can be rewritten in the following form:

$$z = \tan\Psi_s \cdot R - a \cdot (R^2 + z^2)/2, \tag{1.3}$$

where Ψ_s expresses the angle of emission, and R is the horizontal distance. If sound propagates near the ground,

$$z \ll R, \tag{1.4}$$

and the source is above the ground surface ($H_s > 0$), then Eq. (1.3) takes the form of parabola,

$$z = H_s + \tan\Psi_s \cdot R - a \cdot R^2/2. \tag{1.5}$$

A long time ago BARTON [1] has proved that the ray path obeys parabola, when the velocity profile is approximated by linear function. Starting from his result given by Eq. (1.5), we derrived expressions for the ray's vertex, angle of reflection (Sec. 2), shadow zone (Sec. 3), and multiple reflections (Sec. 4).

2. Ray path geometry

For a homogeneous atmosphere with $a = 0$, Eq. (1.5) yields a straight line,

$$z_0(R) = H_s + \tan\Psi_s \cdot R, \tag{2.1}$$

with a constant slope, $\tan\Psi_s$, as expected. For a negative, $a < 0$, and positive value of the equivalent gradient, $a < 0$, the ray leaving a source at the angle, Ψ_s, is bent upward

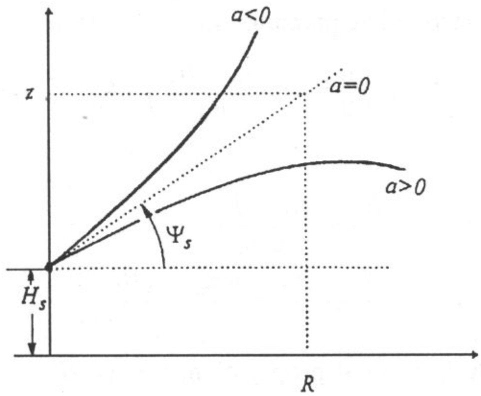

Fig. 2. For negative ($a < 0$) and positive value of the equivalent gradient ($a > 0$), the ray is bending upward and downward, respectively (Eqs. (1.5)–(2.2)).

and downward, respectively (Fig. 2),

$$z(R) = z_0(R) - a \cdot R^2/2. \tag{2.2}$$

The vertex of the parabola defined by Eq. (1.5) occurs at,

$$R_m = \frac{\tan \Psi_s}{a}, \qquad z_m = H_s + \frac{\tan^2 \Psi_s}{2a}. \tag{2.3}$$

Figure 3 shows that both the emission angle and the equivalent gradient are simultaneously negative ($\Psi_s < 0$, $a < 0$) or positive ($\Psi_s > 0$, $a > 0$), so the horizontal distance to the vertex from the source remains positive, $R_m > 0$. The value of z_m determines the ray's height above the ground at the zenith ($R = R_m$).

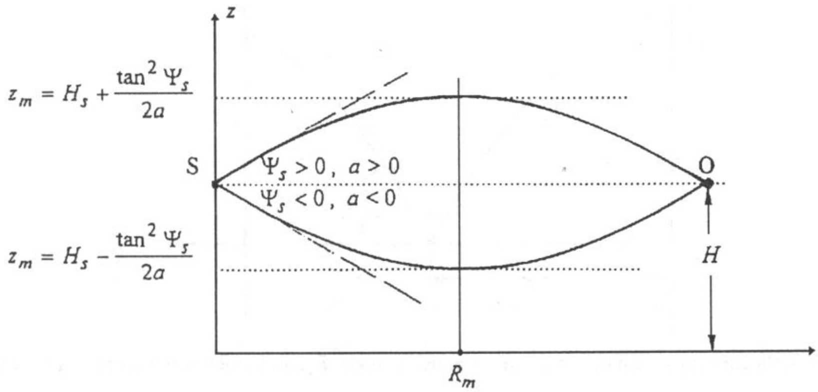

Fig. 3. The ray's height above the ground at the zenith, z_m Eq. (2.3), depends upon the equivalent gradient, a.

The emission angle of a ray, Ψ_s, that reaches the receiver at $R = R_0$ and $z = H_0$, can be calculated from,

$$\tan \Psi_s = \frac{H_0 - H_s}{R_0} + \frac{a}{2} \cdot R_0. \tag{2.4}$$

The equation of the corresponding parabola is,

$$z = H_s + \left(\frac{a}{2}R_0 + \frac{H_0 - H_s}{R_0}\right) \cdot R - \frac{a}{2} \cdot R^2, \tag{2.5}$$

with the vertex at,

$$\begin{aligned} R_m &= \frac{R_0}{2} + \frac{H_0 - H_s}{aR_0}, \\ z_m &= \frac{H_s + H_0}{2} + \frac{aR_0^2}{8} + \frac{(H_0 - H_s)^2}{2aR_0^2}. \end{aligned} \tag{2.6}$$

Considering a nearly horizontal propagation, we set $H_s \approx H_0 \approx H$ (Fig. 3) and obtain,

$$R_m \approx \frac{R_0}{2}, \qquad z_m \approx H + \frac{aR_0^2}{8}. \tag{2.7}$$

It is clear that the vertex occurs halfway between source, S, and receiver, O. For a small value of the equivalent gradient, a, the ray is deflected slightly upward ($z_m > H$) or downward ($z_m < H$), depending upon the sign of a. The situation changes when the value of a becomes greater (See Secs. 3 and 4).

Although it is beyond of the scope of this paper, let's calculate the angle of reflection from the ground surface, Ψ_g (Fig. 4), which is used for the calculation of the sound pressure over a finite impedance ground [7]. When a ray strikes the ground, the angle of reflection can be calculated from the derivative dz/dR (Eq. (1.5)),

$$\tan \Psi_g = \frac{H_s}{R} + \frac{a}{2}R_0. \tag{2.8}$$

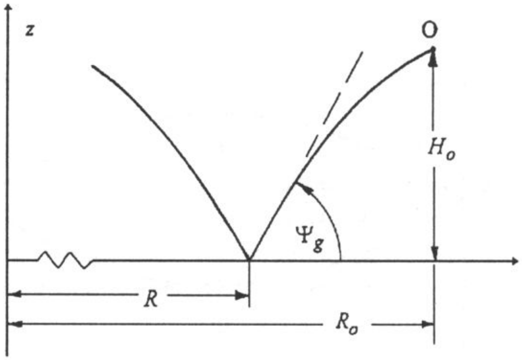

Fig. 4. Reflected ray originates at $(R, 0)$ at emision angle Ψ_g, and reaches the receiver (R_0, H_0).

3. Effect of a negative gradient

For a negative value of the equivalent gradient, $a < 0$, the critical ray may grazes the ground ($\Psi_s = 0$, $z_m = 0$) at some distance, R_*, from the source (Fig. 5). Substituting z

by H_0 in Eq. (1.5) we get equation of the critical ray,

$$R = \sqrt{\frac{2}{-a}}\left(\sqrt{H_s} - \sqrt{z}\right), \qquad 0 \le R \le R_*, \tag{3.1}$$

$$R = \sqrt{\frac{2}{-a}}\left(\sqrt{H_s} + \sqrt{z}\right), \qquad R_* \le R < \infty, \tag{3.2}$$

where (Eq. (2.8)),

$$R_* = \sqrt{\frac{2H_s}{-a}}\,. \tag{3.3}$$

Setting $z_m = 0$ in Eq. (2.3) the emision angle of the critical ray emerges,

$$\tan \Psi_s = -\sqrt{-2aH_s}\,. \tag{3.4}$$

Finally, this result, in conjunction with Eq. (1.5), yields the alternative equation of the critical ray,

$$z = H_s - \sqrt{-2aH_s}\cdot R - \frac{a}{2}\cdot R^2, \qquad 0 \le R < \infty. \tag{3.5}$$

For $R > R_*$ the critrical ray separates the insonified and shadow zones (Fig. 5). To obtain the distance to the shadow zone from the source, we substitute $z = H_0$ into Eq. (3.5),

$$R_0 = \sqrt{\frac{2}{-a}}\left(\sqrt{H_s} + \sqrt{H_0}\right). \tag{3.6}$$

Note, that for $z = H_0$ and $R = R_0$ expression (3.2) yields identical result.

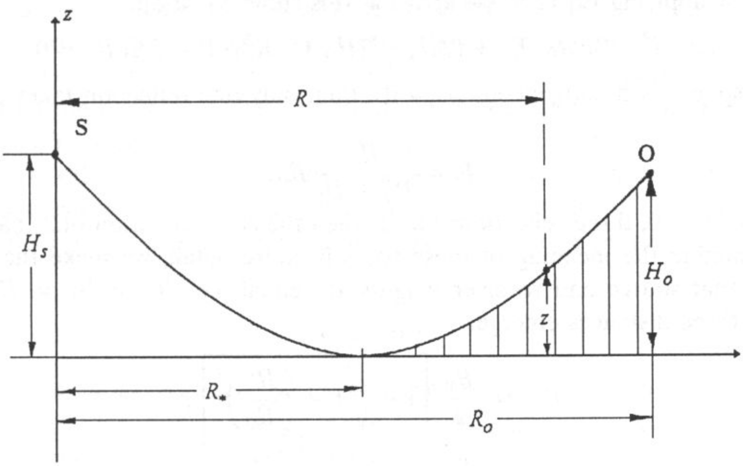

Fig. 5. The boundary of the shadow zone is defined by Eq. (3.6) with $0 \le H_0 < \infty$.

4. Effect of a positive gradient

In the presence of a positive equivalent gradient, in addition to the direct path, there can be paths involving one or more reflections at intermediate points between the source and receiver. As illustrated in Fig. 6, there are three rays that strike the ground at horizontal distances R_1, R_2 and R_3. From elementary geometry one can show that R_1, R_2 and R_3 are determined by roots of the cubic equation.

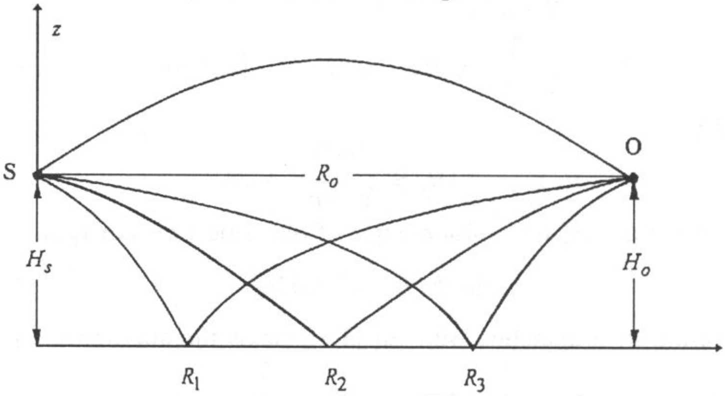

Fig. 6. There are three possible rays that have one reflection at the ground.

The shape of the reflected ray that originates at the source $(0, H_s)$ and reaches the receiver $(R_0, H_0$; see Fig. 4), is given by Eq. (1.5), with the emission angle, Ψ_s, replaced by the angle of reflection Ψ_g (Eq. (2.8)). Setting $z = H_0$, we obtain,

$$H_0 = \tan\Psi_g \cdot (R_0 - R) - \frac{a}{2} \cdot (R_0 - R)^2, \qquad 0 < R < R_0, \tag{4.1}$$

and finally, by applying Eq. (2.8) we arrive at this cubic equation,

$$2a \cdot R^3 - 3aR_0 \cdot R^2 + (2H_s + 2H_0 + aR_0^2) \cdot R - 2H_0R_0 = 0. \tag{4.2}$$

If the atmosphere is homogeneous ($a = 0$), then only one reflection takes place at the distance,

$$R = \frac{H_s}{H_s + H_0} R_0 . \tag{4.3}$$

In the case of $a \neq 0$, three reflections occur when the cubic equation (4.2) has three real roots. To examine the meaning of these roots in more detail, we make the simplifying assumption that source and receiver heights are equal, i.e. $H_s = H_0 = H$. With this assumption three distances emerge,

$$\begin{aligned} R_1 &= \frac{R_0}{2}\left[1 - \sqrt{1 - \left(\frac{R^*}{R_0}\right)^2}\right], \\ R_2 &= \frac{R_0}{2}, \\ R_3 &= \frac{R_0}{2}\left[1 + \sqrt{1 + \left(\frac{R^*}{R_0}\right)^2}\right], \end{aligned} \tag{4.4}$$

where,

$$R^* = 2\sqrt{\frac{2H}{a}} . \tag{4.5}$$

The root R_2 represents the obvious ray that intersects the ground halfway between source and receiver. The other two roots, R_1 and R_3, are symmetrically displaced from R_2. The first corresponds to the ray that strikes the ground near the source ($R = 0$) and the second describes the ray striking the ground near the receiver ($R = R_0$). If the distance between the source and receiver is not sufficiently large, $R_0 < R^*$, i.e.,

$$R_0 < 2\sqrt{\frac{2H}{a}}, \tag{4.6}$$

then the two roots, R_1 and R_3, become complex and only one reflection takes place at $R_2 = R_0/2$, as if the atmosphere were homogeneous ($a = 0$).

A related analysis of this problem for a circular ray path is reported in Ref. [2].

5. Conclusion

Under the assumptions that,

• the atmosphere is stratified, i.e., the sound speed, $\tilde{c}$, and wind speed, V, depend upon the height above the ground,

• the crosswind is ignored, so refraction can be described as if the atmosphere were at rest and characterized by the effective sound speed, $c(z)$ (Eq. (1.1)),

• the effective sound speed is approximated by the linear function (Eq. (1.2)),

the sound ray path has a form of parabola (Eq. (1.5)). For nearly horizontal propagation (Eq. (1.4)), parabola portrays a realistic shape of a ray.

Making use of Eq. (1.5) we have derived the expressions for the ray's vertex (Eqs. (2.3) and (2.6)), angle of reflection (Eq. (2.8)), the border of the shadow zone (Eqs. (3.1), (3.2) and (3.5)), and the horizontal distances for multiple reflections (Eqs. (4.1) – (4.6)).

Corresponding formulae for the ray's vortex, angle of reflection, etc., derived for a circular, or any other path ray, are more complicated than mentioned above equations. Therefore one can say that parabola yields the simplest ray theory.

The results presented in this paper can be useful for noise propagation problems.

References

[1] E.H. Barton, *On the refraction of sound by wind*, Phil. Mag., **1**, 159–165 (1901).

[2] T.F.W. Embleton, G.J. Thiessen and J.E. Piercy, *Propagation in an inversion and reflections of the ground*, J. Acoust. Soc. Am., **59**, 272–282 (1976).

[3] S. Fujiwara, *On the abnormal propagation of sound*, Bulletin Central Meteorological Observatory of Japan, **1**, 2 (1912).

[4] B. Hallberg, C. Larsson and S. Israelsson, *Numerical ray tracing in the atmospheric surface layer*, J. Acoust. Soc. Am., **83**, 2059–2068 (1988).

[5] T. Hidaka, K. Kageyama and S. Masuda, *Sound propagation in the rest atmosphere with linear sound velocity profile*, J. Acoust. Soc. Jpn. (E), **6**, 117–125 (1985).

[6] H. Klug, *Sound speed profiles determined from outdoor sound propagation*, J. Acoust. Soc. Am., **90**, 475–481 (1991).

[7] A. L'ESPERANCE, P. HERZOG, G.A. DAIGLE and J.R. NICOLAS, *Heuristic model for outdoor sound propagation based on an extension of the geometrical ray theory in the case of a linear sound speed profile*, Applied Acoustics, **37**, 111–139 (1992).

[8] R. MAKAREWICZ, *The shadow zone in stratified medium*, J. Acoust. Soc. Am., **85**, 1092–1096 (1989).

[9] L. NYBORG and D. MINZER, *Review of sound propagation in the lower atmosphere*, WADC Technical Report S4-602, 1955.

[10] A.D. PIERCE, *Acoustics*, McGraw Hill, New York 1981.

[11] K.B. RASMUSSEN, *Outdoor sound propagation under the influence of wind and temperature gradient*, J. Sound and Vibr., **104**, 321–335 (1986).

[12] I. RUDNIK, *Propagation of sound in the open air*, [in:] Handbook of Noise Control, C.M. HARRIS [Ed.], McGraw Hill, New York 1957.

[13] W.K. VAN MOORHEM, Y. MA and J.M. BROWN, *Ray paths near the ground in a realistic, thermally stratified atmosphere*, J. Acoust. Soc. Am., **80**, 650–655 (1986).